

Exact Sequences in Non-Exact Categories (An Application to Semimodules)

Jawad Y. Abuhlail*

Department of Mathematics and Statistics
King Fahd University of Petroleum & Minerals
abuhlail@kfupm.edu.sa

Abstract

We consider a notion of *exact sequences* in any – not necessarily exact – pointed category relative to a given (\mathbf{E}, \mathbf{M}) -factorization structure. We apply this notion to introduce and investigate a new notion of exact sequences of semimodules over semirings relative to the canonical image factorization. Several homological results are proved using the new notion of exactness including some restricted versions of the Short Five Lemma and the Snake Lemma opening the door for introducing and investigating *homology objects* in such categories. Our results apply in particular to the variety of commutative monoids extending results in homological varieties to relative homological varieties.

Introduction

Exact sequences and *exact functors* are important tools in Homological Algebra which was developed first in the categories of modules over rings [CE1956] and generalized later to arbitrary Abelian categories (e.g. [Hel1958]). Different sets of axioms characterizing *additive* abstract categories which can be considered – in some sense – *natural home* for exact sequences were developed over time; such categories were called *exact* (e.g. *Buchsbaum-exact categories* [Buc1955], *Quillen-exact categories* [Qui1973]). For these categories, the defining axioms are usually based on a distinguished class of sequences, called an *exact structure*, which is used to define the (short and long) exact sequences in the resulting exact category as well as exact functors between such exact categories. On the other hand, the so-called *Barr-exact categories* [Bar1971], which are *regular categories* with canonical $(\mathbf{RegEpi}, \mathbf{Mono})$ -factorization structures (e.g. [Gri1971], [AHS2004, 14.E], [Bor1994b]), provide an alternative notion of exactness in possibly *non-additive* categories. In such categories, the role of exact sequences is played by the so-called *exact forks* which are also

*The author would like to acknowledge the support provided by the Deanship of Scientific Research (DSR) at King Fahd University of Petroleum & Minerals (KFUPM) for funding this work through project No. FT100004.

used to define exact functors between Barr-exact categories. For a systematic study and comprehensive exposition of these and other notions of exact categories, the interested reader is advised to consult [Bue2010].

An elegant notion of exact categories to which we refer often in this manuscript is due to Puppe [Pup1962] (see also Mitchell [Mit1965]). We call a category \mathfrak{C} a *Puppe-exact category* iff it is pointed (i.e. $\text{Hom}_{\mathfrak{C}}(A, B)$ has a zero morphism for each $A, B \in \text{Obj}(\mathfrak{C})$) and has a **(NormalEpi, NormalMono)**-factorization structure (e.g. [AHS2004, 14.F]); such a category is *additive* if and only if it is Abelian (cf. [BP1969, 3.2]). By [Sch1972, 13.1.3], any Puppe-exact category has kernels and cokernels; moreover it is *normal* (i.e. every monomorphism is a kernel) and *conormal* (i.e. every epimorphism is a cokernel). The *image* (*coimage*) of a morphism γ in a Puppe-exact category \mathfrak{C} is defined as $\text{Im}(\gamma) := \text{Ker}(\text{coker}(\gamma))$ ($\text{Coim}(\gamma) := \text{Coker}(\text{ker}(\gamma))$) and a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in \mathfrak{C} is said to be *exact* iff $\text{Im}(f) \simeq \text{Ker}(g)$ or equivalently $\text{Coim}(g) \simeq \text{Coker}(f)$ [Sch1972, 12.4.9, 13.1.3].

Many interesting pointed categories are not Puppe-exact, (e.g. some varieties of Universal Algebra like the variety **Grp** of groups, the variety **Mon** of monoids and the variety **pSet** of pointed sets). Thus, the following question arises naturally:

Question: *When is an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in a pointed category exact?*

The main goal of this article is providing an answer to the above mentioned question. Our approach is based on analyzing the notion of exact sequences in Puppe-exact categories and then generalizing it to any pointed category \mathfrak{C} relative to a given **(E, M)**-factorization structure, which always exists [AHS2004, Section 14] (see also [Bar2002]): we say that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **(E, M)**-exact iff there exist $f' \in \mathbf{E}$ and $g'' \in \mathbf{M}$ such that $f = \text{ker}(g) \circ f'$ and $g = g'' \circ \text{coker}(f)$ are the *essentially unique* **(E, M)**-factorizations of f and g in \mathfrak{C} . To illustrate this notion of exactness, we introduce a restricted version of the Short Exact Lemma and introduce a class of *relative homological categories* which generalizes the notion of *homological categories* in the sense of Borceux and Bourn [BB2004, Chapter 4].

Before we proceed, we find it suitable to include the following clarification. A successful notion of exact sequences already exists in several pointed categories which are not Puppe-exact (e.g. in **Grp**): A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of groups is exact iff $\text{Im}(f) \simeq \text{Ker}(g)$. While used in many papers, and even considered *standard*, this notion of exactness is not necessarily appropriate in other pointed categories (e.g. in **Mon**). We briefly demonstrate why we believe this is the case. Firstly, one should be careful about the definition of the image (coimage) of a morphisms γ in a category which is not Puppe-exact: although several authors define $\text{Im}(\gamma) := \text{Ker}(\text{coker}(\gamma))$ ($\text{Coim}(\gamma) := \text{Coker}(\text{ker}(\gamma))$), this might not be the appropriate notion in a category which is not Puppe-exact as it does not necessarily satisfy the universal property that an image (coimage) is supposed to satisfy (cf. [Fai1973, 5.8.7] and [EW1987]). Secondly, even if the appropriate image (coimage) is used, one has to take into consideration the natural dual condition of exactness, namely $\text{Coim}(g) \simeq \text{Coker}(f)$. This *hidden* condition is equivalent to $\text{Im}(f) \simeq \text{Ker}(g)$ in Puppe-exact categories [Sch1972, Lemma 13.1.4]; however, this is not necessarily the case in categories which are not Puppe-exact. So, one might end up with two different notions: *left-exact sequences* for which $\text{Im}(f) \simeq \text{Ker}(g)$ and *right-exact sequences* for which $\text{Coim}(g) \simeq \text{Coker}(f)$, while exact

sequences have to be defined as those which are left-exact and right-exact.

An example that demonstrates how adopting the definition of exact sequences in Puppe-exact categories to arbitrary pointed categories might create serious problems is the notion of exact sequences of semimodules over semirings due to Takahashi [Tak1981]. An unfortunate choice of a notion of exactness and an inappropriate choice of a *tensor functor* which is not left adjoint of the Hom functor, in addition to the *bad* nature of monoids (in contrast with the *good* nature of groups), are among the main reasons for failing to develop a satisfactory homological theory for semimodules or commutative monoids so far (there are indeed many successful investigations related to the homology of monoids, e.g. [KKM2000]).

This manuscript is divided as follows. After this introduction, and for the convention of the reader, we recall in Section 1 some terminology and notions from Category Theory. In particular, we analyze the notion of exact sequences in *Puppe-exact categories* and use that analysis to introduce a new notion of exact sequences in arbitrary pointed categories. Moreover, we present some special classes of morphisms which play an important role in the sequel. In Section 2, we collect some definitions and results on semirings and semimodules and clarify the differences between the terminology used in this paper and the classical terminology; we also clarify the reason for changing some terminology. In Section 3, we apply our general definition of exactness to obtain a new notion of exact sequences of semimodules over semirings. We demonstrate how this notion enables us to characterize in a very simple way, similar to that in homological categories, different classes of morphisms (e.g. monomorphisms, regular epimorphisms, isomorphisms). In Section 4, we illustrate the advantage of our notion of exactness over the existing ones by showing how it enables us to prove some of the elementary diagram lemmas for semimodules over semirings. Moreover, we introduce a restricted version of the *Short Five Lemma* 4.7, which characterizes the homological categories among the pointed regular ones, and use it to introduce a new class of *relative homological categories* w.r.t. a given factorization structure and a special class of morphisms. The category of cancellative semimodules over semirings, in particular the category of cancellative commutative monoids, is introduced as a prototype of such categories. Moreover, we prove a restricted version of the *Snake Lemma* 4.13 for cancellative semimodules (cancellative commutative monoids) which opens the door for introducing and investigating *homology objects* in such categories.

1 Exact Sequences in Pointed Categories

Throughout, and unless otherwise explicitly mentioned, \mathfrak{C} is an arbitrary *pointed* category (i.e. $\text{Hom}_{\mathfrak{C}}(A, B)$ has a zero morphism); all objects and morphisms are assumed to be in \mathfrak{C} . When clear from the context, we may drop \mathfrak{C} . Our main references in Category Theory are [AHS2004] and [Mac1998].

1.1. A *monomorphism* in \mathfrak{C} is a morphism m such that for any morphisms f_1, f_2 :

$$m \circ f_1 = m \circ f_2 \Rightarrow f_1 = f_2.$$

An *equalizer* of a family of morphisms $(f_\lambda : A \rightarrow B)_\Lambda$ in \mathfrak{C} is a morphism $g : A' \rightarrow A$ in \mathfrak{C} such that $f_\lambda \circ g = f_{\lambda'} \circ g$ for all $\lambda, \lambda' \in \Lambda$ and whenever there exists $g' : A'' \rightarrow A$ with

$f_\lambda \circ g' = f_{\lambda'} \circ g'$ for all $\lambda, \lambda' \in \Lambda$ then there exists a *unique* morphism $\tilde{g} : A'' \rightarrow A'$ such that $g \circ \tilde{g} = g'$:

$$\begin{array}{ccccc} & & A'' & & \\ & \swarrow \tilde{g} & \downarrow g' & \searrow f_\lambda & \\ A' & \xrightarrow{g} & A & \xrightarrow{\quad} & B \end{array}$$

With $\text{Equ}((f_\lambda)_{\lambda \in \Lambda})$ we denote the domain of the *essentially unique* equalizer of $(f_\lambda)_{\lambda \in \Lambda}$, if it exists. A morphism g in \mathfrak{C} is said to be a *regular monomorphism* iff $g = \text{equ}(f_1, f_2)$ for two morphisms f_1, f_2 in \mathfrak{C} .

1.2. Let g be a morphism in \mathfrak{C} . We call $\ker(f) := \text{Equ}(f, 0)$ the *kernel* of f . We say that g is a *normal monomorphism* iff $g = \ker(f)$ for some morphism f in \mathfrak{C} . The category \mathfrak{C} is said to be *normal* iff every monomorphism in \mathfrak{C} is normal.

1.3. An *epimorphism* in \mathfrak{C} is a morphism e such that for any morphisms f_1, f_2 :

$$f_1 \circ e = f_2 \circ e \Rightarrow f_1 = f_2.$$

A coequalizer of a family of morphisms $(f_\lambda : A \rightarrow B)_\Lambda$ in \mathfrak{C} is a morphism $g : B \rightarrow B'$ in \mathfrak{C} such that $g \circ f_\lambda = g \circ f_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda$ and whenever there exists $g' : B \rightarrow B''$ with $g' \circ f_\lambda = g' \circ f_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda$ then there exists a *unique* morphism $\tilde{g} : B' \rightarrow B''$ such that $\tilde{g} \circ g = g'$:

$$\begin{array}{ccccc} A & \xrightarrow{f_\lambda} & B & \xrightarrow{g} & B' \\ & & \downarrow g' & \swarrow \tilde{g} & \\ & & B'' & & \end{array}$$

With $\text{Coequ}((f_\lambda)_{\lambda \in \Lambda})$ we denote the codomain of the essentially unique coequalizer of $(f_\lambda)_{\lambda \in \Lambda}$, if it exists. A morphism g is said to be a *regular epimorphism* iff $g = \text{Coequ}(f_1, f_2)$ for two morphisms f_1, f_2 in \mathfrak{C} .

1.4. We call $\text{Coker}(f) := \text{Coequ}(f, 0)$ the *cokernel* of f . A morphism g is said to be a *conormal epimorphism* iff $g = \text{coker}(f)$ for some morphism f in \mathfrak{C} . The category \mathfrak{C} is said to be *conormal* iff every epimorphism in \mathfrak{C} is conormal.

Notation. We fix some notation:

- With **Mono**(\mathfrak{C}) (**RegMono**(\mathfrak{C})) we denote the class of (regular) monomorphisms in \mathfrak{C} and by **Epi**(\mathfrak{C}) (**RegEpi**(\mathfrak{C})) the class of (regular) epimorphisms in \mathfrak{C} . We denote by **NormMono**(\mathfrak{C}) \subseteq **RegMono**(\mathfrak{C}) (**NormEpi**(\mathfrak{C}) \subseteq **RegEpi**(\mathfrak{C})) the class of normal monomorphisms (normal epimorphisms) in \mathfrak{C} .
- With **Iso**(\mathfrak{C}) we denote the class of isomorphisms and with **Bimor**(\mathfrak{C}) the class of bimorphisms (i.e. monomorphisms and epimorphisms) in \mathfrak{C} .
- Let \mathfrak{C} be concrete (over the category **Set** of sets) with underlying functor $U : \mathfrak{C} \rightarrow \mathbf{Set}$. We denote by **Inj**(\mathfrak{C}) (**Surj**(\mathfrak{C})) the class of morphisms γ in \mathfrak{C} such that $U(\gamma)$ is an injective (surjective) map.

Remark 1.5. (e.g. [AHS2004, 7.76]) We have

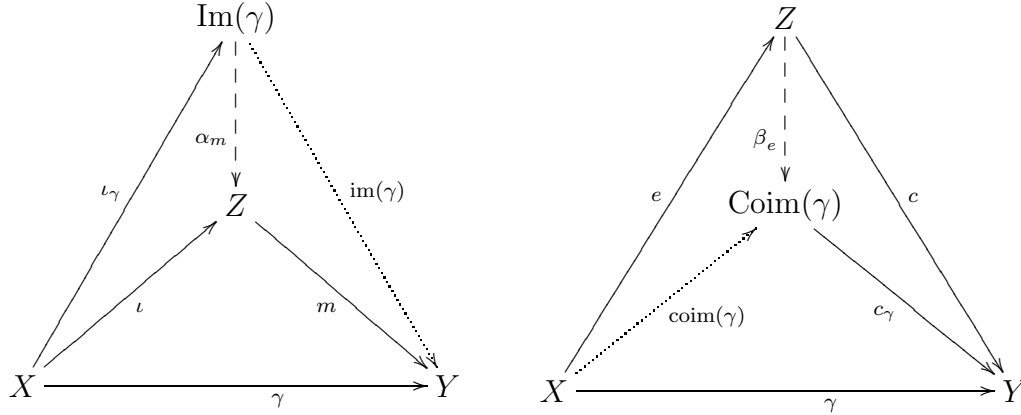
$$\mathbf{Iso}(\mathfrak{C}) \subseteq \mathbf{NormMono}(\mathfrak{C}) \subseteq \mathbf{RegMono}(\mathfrak{C}) \subseteq \mathbf{Mono}(\mathfrak{C})$$

and

$$\mathbf{Iso}(\mathfrak{C}) \subseteq \mathbf{NormEpi}(\mathfrak{C}) \subseteq \mathbf{RegEpi}(\mathfrak{C}) \subseteq \mathbf{Epi}(\mathfrak{C}).$$

Definition 1.6. (Compare with [Fai1973, 5.8.7], [EW1987], [Bar2002]) Let \mathbf{E} and \mathbf{M} be classes of morphisms in \mathfrak{C} and $\gamma : X \longrightarrow Y$ a morphism in \mathfrak{C} .

1. The \mathbf{M} -*image* of γ is $\text{im}(\gamma) : \text{Im}(\gamma) \longrightarrow Y$ in \mathbf{M} such that $\gamma = \text{im}(\gamma) \circ \iota_\gamma$ for some morphism ι_γ and if $\gamma = m \circ \iota$ for some $m \in \mathbf{M}$, then there exists a unique morphism $\alpha_m : \text{Im}(\gamma) \longrightarrow Z$ such that $m \circ \alpha_m = \text{im}(\gamma)$ and $\alpha_m \circ \iota_\gamma = \iota$.
2. The \mathbf{E} -*coimage* of γ is $\text{coim}(\gamma) : X \longrightarrow \text{Coim}(\gamma)$ in \mathbf{E} such that $\gamma = c_\gamma \circ \text{coim}(\gamma)$ for some morphism c_γ and if $\gamma = c \circ e$ for some $e \in \mathbf{E}$, then there exists a unique morphism $\beta_e : Z \longrightarrow \text{Coim}(\gamma)$ such that $\beta_e \circ e = \text{coim}(\gamma)$ and $c_\gamma \circ \beta_e = c$.



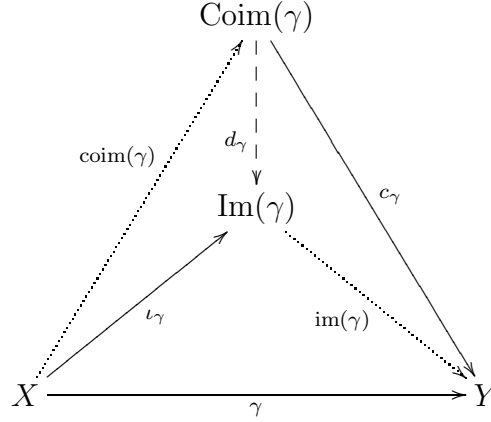
Definition 1.7. ([AHS2004, 14.1]) Let \mathbf{E} and \mathbf{M} be classes of morphisms in \mathfrak{C} . The pair (\mathbf{E}, \mathbf{M}) is called a *factorization structure* (for morphisms in) \mathfrak{C} , and \mathfrak{C} is said to be (\mathbf{E}, \mathbf{M}) -structured, provided that

1. \mathbf{E} and \mathbf{M} are closed under composition with isomorphisms.
2. \mathfrak{C} has (\mathbf{E}, \mathbf{M}) -factorizations, i.e. each morphism f in \mathfrak{C} has a factorization $f = m \circ e$ with $m \in \mathbf{M}$ and $e \in \mathbf{E}$.
3. \mathfrak{C} has the *unique (\mathbf{E}, \mathbf{M}) -diagonalization property* (or the *diagonal-fill-in property*) i.e. for each commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ f \downarrow & \swarrow d & \downarrow g \\ C & \xrightarrow{m} & D \end{array}$$

with $e \in \mathbf{E}$ and $m \in \mathbf{M}$, there exists a *unique* morphism $d : B \longrightarrow C$ such that $d \circ e = f$ and $m \circ d = g$.

1.8. Let \mathfrak{C} be an (\mathbf{E}, \mathbf{M}) -structured category and $\gamma : X \longrightarrow Y$ be a morphism in \mathfrak{C} with (\mathbf{E}, \mathbf{M}) -factorization $\gamma : X \xrightarrow{e} U \xrightarrow{m} Y$. Let $\text{Coim}(\gamma)$ and $\text{Im}(\gamma)$ be the the \mathbf{E} -coimage of γ and the \mathbf{M} -image of γ , respectively. Then there exist isomorphisms $\text{Coim}(\gamma) \xrightarrow{d_1} U \xrightarrow{d_2} \text{Im}(\gamma)$ such that $d_2 \circ d_1$ is the canonical morphism $d_\gamma : \text{Coim}(\gamma) \longrightarrow \text{Im}(\gamma)$, which is in this case an isomorphism:



- Remarks 1.9.*
1. For any category, $(\mathbf{Iso}, \mathbf{Mor})$ and $(\mathbf{Mor}, \mathbf{Iso})$ are *trivial* factorization structures.
 2. Some authors assume that $\mathbf{E} \subseteq \mathbf{Epi}(\mathfrak{C})$ and $\mathbf{M} \subseteq \mathbf{Mon}(\mathfrak{C})$ (e.g. [Bar2002]).
 3. If (\mathbf{E}, \mathbf{M}) is a factorization structure for \mathfrak{C} , then $\mathbf{E} \cap \mathbf{M} = \mathbf{Iso}(\mathfrak{C})$.
 4. As a result of the unique diagonalization property, any (\mathbf{E}, \mathbf{M}) -factorization in an (\mathbf{E}, \mathbf{M}) -structured category is *essentially* unique (compare with [AHS2004, Proposition 14.4]). Suppose that $m_1 \circ e_1 = \gamma = m_2 \circ e_2$ are two (\mathbf{E}, \mathbf{M}) -factorizations of a morphism $\gamma : A \longrightarrow B$ in \mathfrak{C}

$$\begin{array}{ccc} A & \xrightarrow{e_1} & C_1 \\ e_2 \downarrow & \swarrow h & \downarrow m_1 \\ C_2 & \xrightarrow{m_2} & B \end{array}$$

Then there exists a (unique) isomorphism $h : C_1 \longrightarrow C_2$ s.t. the above diagram commutes.

Exact Categories

There are several notions of *exact sequences* and *exact categories* in the literature (e.g. [Buc1955], [Qui1973], [Pup1962], [Bar1971]).

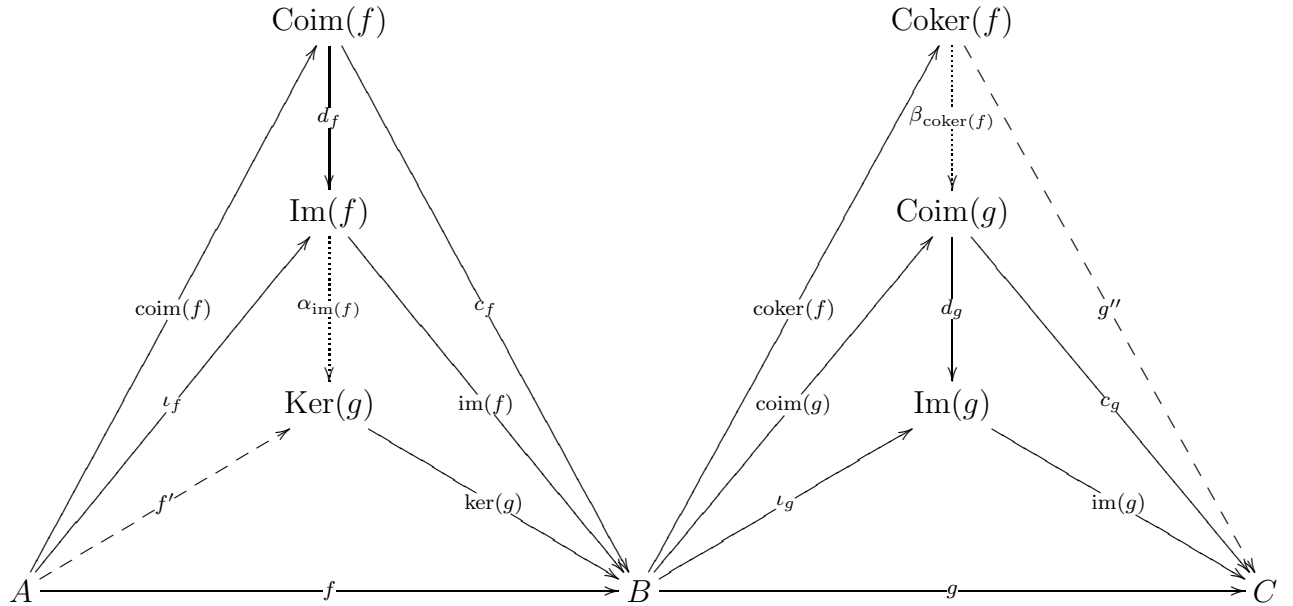
1.10. Call \mathfrak{C} a *Puppe-exact category* iff it is pointed and has a $(\mathbf{NormalEpi}, \mathbf{NormalMono})$ -factorization structure. By [AHS2004, 14.F (a)], a pointed category is Puppe-exact if and only if it has $(\mathbf{NormalEpi}, \mathbf{NormalMono})$ -factorizations, i.e. every morphism γ admits a – necessarily *unique* – factorization $\gamma = \gamma'' \circ \gamma'$ such that γ' is a cokernel and γ'' is a kernel. The image and the coimage of a morphism $\gamma : X \longrightarrow Y$ in such a categories are given by $\text{Im}(\gamma) := \text{Ker}(\text{coker}(\gamma))$ and $\text{Coim}(\gamma) := \text{Coker}(\text{ker}(\gamma))$, respectively. Moreover, a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be *exact* iff $\text{Im}(f) \simeq \text{Ker}(g)$.

Remarks 1.11. 1. Every non-empty Puppe-exact category has a zero-object, kernels and cokernels, is normal, conormal and has equalizers.

2. Let \mathfrak{C} be a category with a zero-object, kernels, cokernels and equalizers. If \mathfrak{C} is normal, then $\text{Coker}(\ker(\gamma)) \simeq \text{Ker}(\text{coker}(\gamma))$ for any morphism γ in \mathfrak{C} [Fai1973, Proposition 5.20].

In light of the previous remarks, [Sch1972, Lemma 31.14] can be restated as follows:

Lemma 1.12. *Let \mathfrak{C} be a Puppe-exact category, $A \xrightarrow{f} B \xrightarrow{g} C$ a sequence in \mathfrak{C} with $g \circ f = 0$ and consider the following commutative diagram with canonical and induced factorizations*



The following are equivalent:

1. $A \xrightarrow{f} B \xrightarrow{g} C$ is exact (i.e. $\text{Im}(f) \xrightarrow{\alpha_{\text{im}(f)}} \text{Ker}(g)$);
2. $\text{Coker}(f) \xrightarrow{\beta_{\text{coker}(f)}} \text{Coim}(g)$;
3. $\text{Coim}(f) \simeq \text{Ker}(g)$;
4. $\text{Coker}(f) \simeq \text{Im}(g)$;
5. $\text{Im}(f) \simeq \text{Ker}(\text{coim}(g))$;
6. $\text{Coim}(g) \simeq \text{Coker}(\text{im}(f))$;
7. f' is a (normal) epimorphism;
8. g'' is a (normal) monomorphism.

Inspired by the previous lemma, we introduce a notion of exact sequences in any pointed category:

Definition 1.13. Let \mathfrak{C} be any pointed category and fix a factorization structure (\mathbf{E}, \mathbf{M}) for \mathfrak{C} . We call a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \quad (1)$$

exact w.r.t. (\mathbf{E}, \mathbf{M}) iff f and g have factorizations

$$f = \ker(g) \circ f' \text{ and } g = g'' \circ \operatorname{coker}(f) \text{ with } (f', \ker(g)), (\operatorname{coker}(f), g'') \in \mathbf{E} \times \mathbf{M}.$$

$$\begin{array}{ccccc} & & A & & \\ & \swarrow f' & \downarrow f & & \\ \operatorname{Ker}(g) & \xrightarrow{\ker(g)} & B & \xrightarrow{\operatorname{coker}(f)} & \operatorname{Coker}(f) \\ & & \downarrow g & \swarrow g'' & \\ & & C & & \end{array}$$

When the factorization structure is clear from the context we drop it. We call a sequence

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

exact at A_i iff $A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1}$ is exact; moreover, we call this sequence *exact* iff it is exact at A_i for every i . An exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad (2)$$

is called a *short exact sequence*.

Remark 1.14. Let \mathfrak{C} be a pointed category and fix a factorization structure (\mathbf{E}, \mathbf{M}) for \mathfrak{C} . It follows immediately from the definition that (2) is a short exact sequence w.r.t. (\mathbf{E}, \mathbf{M}) if and only if $\operatorname{Coker}(f) \in \mathbf{E}$, $\operatorname{Ker}(g) \in \mathbf{M}$, $f = \operatorname{Ker}(g)$ and $g = \operatorname{Coker}(f)$.

Example 1.15. Let \mathfrak{C} be a Puppe exact category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact* if and only if $f = \ker(g) \circ f'$, $g = g'' \circ \operatorname{coker}(f)$ with $(f', \ker(g)), (\operatorname{coker}(f), g'') \in \mathbf{NormalEpi} \times \mathbf{NormalMono}$. Notice that, by Lemma 1.12, this is equivalent to the classical notion of exacts sequences in Puppe-exact categories, namely $\operatorname{Im}(f) \simeq \operatorname{Ker}(g)$. This applies in particular to the categories of modules over rings (e.g. the category \mathbf{Ab} of Abelian groups).

Example 1.16. Let \mathfrak{C} be a pointed (\mathbf{E}, \mathbf{M}) -structure category with $\mathbf{NormalEpi} \subseteq \mathbf{E} \subseteq \mathbf{Epi}$ and $\mathbf{NormalMono} \subseteq \mathbf{M} \subseteq \mathbf{Mono}$. Then a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in \mathfrak{C} is exact if and only if the essentially unique (\mathbf{E}, \mathbf{M}) factorizations $f = m_1 \circ e_1$, $g = m_2 \circ e_2$ can be chosen so that $m_1 = \ker(e_2)$ and $e_2 = \operatorname{coker}(m_1)$. Moreover, a sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if $f = \operatorname{Ker}(g)$ and $g = \operatorname{Coker}(f)$. This applies to general pointed categories which are $(\mathbf{RegEpi}, \mathbf{Mono})$ -structured or $(\mathbf{Epi}, \mathbf{RegMono})$ -structured. In particular, this applies to pointed regular categories (compare with [BB2004, Definition 4.1.7]).

Example 1.17. Let \mathfrak{C} be a pointed protomodular category (in the sense of D. Bourn [Bou1991]) with finite limits. By [BB2004, Proposition 3.1.23], $g \in \mathbf{RegEpi}(\mathfrak{C})$ if and only if $g = \text{coker}(\ker(g))$. If \mathfrak{C} is $(\mathbf{RegEpi}, \mathbf{Mono})$ -structured or $(\mathbf{Epi}, \mathbf{RegMono})$ -structured, then it follows that a sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ in \mathfrak{C} is exact if and only if $f = \ker(g)$ and g is a regular epimorphism. This applies in particular to homological categories, which are precisely pointed and protomodular regular categories [BB2004].

Example 1.18. Let $(\mathfrak{C}; \mathbf{E})$ be a *relative homological category* in the sense of [Jan2006], where \mathbf{E} is a distinguished class of normal epimorphisms and assume that \mathfrak{C} is $(\mathbf{E}, \mathbf{Mono})$ -structured (which is not actually assumed in the defining axioms of such categories). Analyzing Condition (a) on g' (page 192), which was assumed to prove the so called *Relative Snake Lemma*, shows that this assumption and along with the assumptions on f' are essentially equivalent to assuming that the row $0 \rightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \rightarrow 0$ is $(\mathbf{E}, \mathbf{Mono})$ -exact.

Steady Morphisms

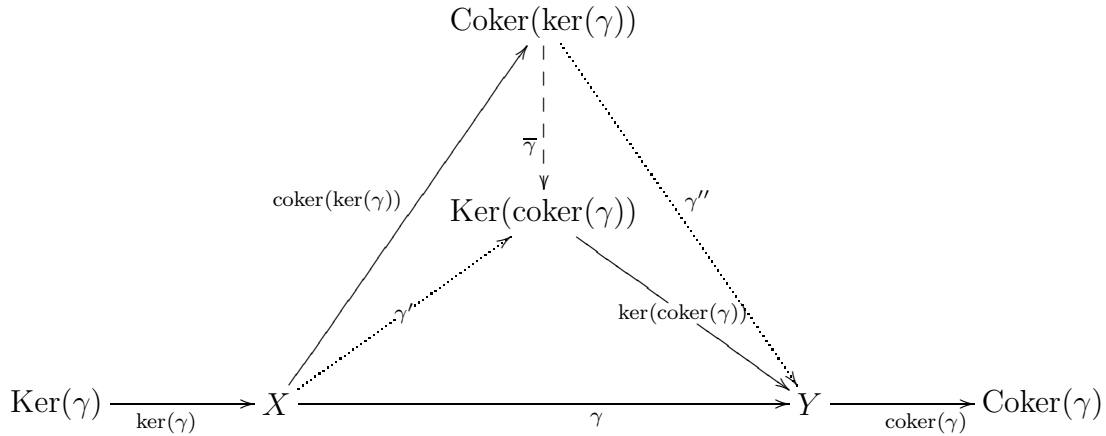
In what follows, we consider a special class of categories to which there is a natural transfer of the notion of exact sequences in Puppe-exact categories.

Definition 1.19. Let \mathfrak{C} be a pointed (\mathbf{E}, \mathbf{M}) -structured category. We say that a morphism $\gamma : X \rightarrow Y$ in \mathfrak{C} is:

steady w.r.t. (\mathbf{E}, \mathbf{M}) iff $\text{Ker}(\gamma)$, $\text{Coker}(\ker(\gamma))$ exist in \mathfrak{C} and γ admits an (\mathbf{E}, \mathbf{M}) -factorization $\gamma = \gamma'' \circ \text{coker}(\ker(\gamma))$, equivalently $\text{Coker}(\ker(\gamma)) \simeq \text{Coim}(\gamma)$;

costeady w.r.t. (\mathbf{E}, \mathbf{M}) iff $\text{Coker}(\gamma)$, $\text{Ker}(\text{coker}(\gamma))$ exist in \mathfrak{C} and γ admits an (\mathbf{E}, \mathbf{M}) -factorization $\gamma = \ker(\text{coker}(\gamma)) \circ \gamma'$, equivalently $\text{Ker}(\text{coker}(\gamma)) \simeq \text{Im}(\gamma)$;

bisteady w.r.t. (\mathbf{E}, \mathbf{M}) iff γ is steady and costeady w.r.t. (\mathbf{E}, \mathbf{M}) , equivalently $\text{Coker}(\ker(\gamma)) \simeq \text{Coim}(\gamma) \xrightarrow{d_\gamma} \text{Im}(\gamma) \simeq \text{Ker}(\ker(\gamma))$.



We call \mathfrak{C} *steady* (resp. *costeady*, *bisteady*) w.r.t. (\mathbf{E}, \mathbf{M}) iff all morphisms in \mathfrak{C} are steady (resp. costeady, bisteady) w.r.t. (\mathbf{E}, \mathbf{M}) .

Remark 1.20. Let \mathfrak{C} be a pointed (\mathbf{E}, \mathbf{M}) -structured category. If \mathfrak{C} is bisteady w.r.t. (\mathbf{E}, \mathbf{M}) , then \mathfrak{C} is Puppe-exact: in this case, every morphism in \mathfrak{C} has a $(\mathbf{NormalEpi}, \mathbf{NormalMono})$ -factorization, whence \mathfrak{C} is Puppe-exact [AHS2004, 14.F]. Moreover, if $\mathbf{NormalEpi} \subseteq \mathbf{E}$ and $\mathbf{NormalMono} \subseteq \mathbf{M}$ then \mathfrak{C} is bisteady w.r.t. (\mathbf{E}, \mathbf{M}) if and only if \mathfrak{C} is Puppe-exact.

1.21. All varieties – in the sense of Universal Algebra – are **(RegEpi, Mono)**-structured. Moreover, the class of regular epimorphisms coincides with that of surjective morphisms, and the class of monomorphisms coincides with that of injective morphisms. Let \mathcal{V} be a pointed variety. We say that a morphism $\gamma : X \rightarrow Y$ in \mathcal{V} is *steady* (resp. *costeady*, *bisteady*) iff γ is steady (resp. costeady, bisteady) w.r.t. **(Surj, Inj)**. With $\text{Im}(\gamma)$ ($\text{Coim}(\gamma)$) we mean the **Inj**-image (the **Surj**-coimage) of γ . Moreover, we say that a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{V} is *exact* iff it is **(Surj, Inj)**-exact.

Example 1.22. The variety **Grp** of all (Abelian and non-Abelian) groups is steady. Let $\gamma : X \rightarrow Y$ be any morphism of groups. Notice that $\text{Ker}(\gamma) = \{x \in X \mid \gamma(x) = 1_Y\}$ while $\text{Coker}(\gamma) = Y/N_\gamma$, where N_γ is the *normal closure* of $\gamma(X)$. Consider the canonical **(Surj, Inj)**-factorization $\gamma : X \xrightarrow{\text{im}(\gamma)} \gamma(X) \xrightarrow{\iota} Y$ where ι is the canonical embedding. Consider also the factorization $\gamma : X \xrightarrow{\text{coker}(\text{ker}(\gamma))} X/\text{Ker}(\gamma) \xrightarrow{\gamma''} Y$. Clearly, γ is steady if and only if γ'' is injective. Indeed, if $\gamma''([x_1]) = \gamma''([x_2])$ for some $x_1, x_2 \in X$, then $\gamma(x_1) = \gamma(x_2)$ whence $\gamma(x_1^{-1}x_2) = 1_Y$ and it follows that $x_1^{-1}x_2 = k$ for some $k \in \text{Ker}(\gamma)$, i.e. $[x_1] = [x_2]$. Consequently, $\gamma'' \in \mathbf{Inj}$. On the other hand, consider the factorization $\gamma : X \xrightarrow{\gamma'} N_\gamma \xrightarrow{\text{ker}(\text{coker}(\gamma))} Y$. Then γ is costeady if and only if $\gamma(X) = N_\gamma$ if and only if $\gamma(X) \leq G$ is a normal subgroup. Clearly, **Grp** is not costeady: Let G be a group, H a subgroup that is not normal in G and let $\gamma : H \hookrightarrow G$ be the embedding. Indeed, $H = \gamma(H) \neq N_\gamma$, i.e. γ is not costeady. Consequently, **Grp** is not a bisteady category.

In the following example, we demonstrate how the *classical* notion of exact sequences of groups is consistent with our new definition of exact sequences in arbitrary pointed categories.

Example 1.23. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of groups and consider the canonical factorizations of $f : A \xrightarrow{\text{im}(f)} f(A) \xrightarrow{\iota_1} B$ and $g : B \xrightarrow{\text{im}(g)} g(B) \xrightarrow{\iota_2} C$. If the given sequence is exact, then $f = \text{ker}(g) \circ f'$ with f' surjective. This implies that $f(A) = \text{Ker}(g)$. On the other hand, assume that $f(A) = \text{Ker}(g)$. Then f has a **(Inj, Surj)**-factorization as $f = \text{ker}(g) \circ \text{im}(f)$. Moreover, it is evident that there is an isomorphism of groups $B/\text{Ker}(g) \cong g(B)$. So, g has an **(Inj, Surj)**-factorization $g = (\iota_2 \circ \gamma) \circ \text{coker}(g)$. It follows that $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $f(A) = \text{Ker}(g)$.

2 Semirings and Semimodules

Semirings (semimodules) are roughly speaking, rings (modules) without subtraction. Semirings were studied independently by several algebraists, especially by H. S. Vandiver [Van1934] who considered them as the *best* algebraic structures which unify rings and bounded distributive lattices. Since the sixties of the last century, semirings were shown to have significant applications in several areas as Automata Theory (e.g. [Eil1974], [Eil1976], [KS1986]), Theoretical Computer Science (e.g. [HW1998]) and many classical areas of mathematics (e.g. [Gol1999a], [Gol1999b]).

Recently, semirings played an important role in several emerging areas of research like Idempotent Analysis (e.g. [KM1997], [LMS2001], [Lit2007]), Tropical Geometry (e.g.

[R-GST2005], [Mik2006]) and many aspects of modern Mathematics and Mathematical Physics (e.g. [Gol2003], [LM2005]). In his dissertation [Dur2007], N. Durov demonstrated that semirings are in one-to-one correspondence with what he called *algebraic additive monads* on the category **Set** of sets. Moreover, a connection between semirings and the so-called \mathbb{F} -rings, where \mathbb{F} is the field with one element, was pointed out in [PL, 1.3 – 1.4].

The theory of semimodules was developed mainly by M. Takahashi, who published several fundamental papers on this topic (cf. [Tak1979] – [Tak1985]) and to whom research in the theory of semimodules over semirings is indeed indebted. However, it seems to us that there are some gaps in his theory of semimodules which led to confusion and sometimes conceptual misunderstandings. Instead of introducing appropriate definitions and notions that fit well with the special properties of the category of semimodules over semirings, some definitions and notions which are fine in *Puppe-exact categories* in general, and in categories of modules over rings in particular, were enforced in a category which is, in general, far away from being Puppe-exact.

A systematic development of the homological theory of semirings and semimodules has been initiated recently in a series of papers by Y. Katsov [Kat1997] and carried out in a continuing series of papers (e.g. [Kat2004a], [Kat2004b], [KTN2009], [KN2011], [IK2011], [IK2011]). Another approach that is worth mentioning was initiated by A. Patchkoria in [Pat1998] and continued in a series of papers (e.g. [Pat2000a], [Pat2000b], [Pat2003], [Pat2006], [Pat2009]).

In what follows, we revisit the category of semimodules over a semiring. In particular, we adopt a new definition of exact sequences of semimodules and investigate it. We also introduce some terminology that will be needed in the sequel.

2.1. Let $(S, *)$ be a semigroup. We call $s \in S$ *cancellable* iff for any $s_1, s_2 \in S$:

$$s_1 * s = s_2 * s \implies s_1 = s_2 \text{ and } s * s_1 = s * s_2 \implies s_1 = s_2.$$

We call S *cancellative* iff all elements of S are cancellable. We say that a morphism of semigroups $f : S \longrightarrow S'$ is *cancellative* iff $f(s) \in S'$ is cancellable for every $s \in S$. We call S an *idempotent semigroup* iff $s * s = s$ for every $s \in S$.

2.2. Let $(S, +)$ be an Abelian additive semigroup. A subset $X \subseteq S$ is said to be *subtractive* iff for any $s \in S$ and $x \in X$ we have: $s + x \in X \implies s \in X$. The *subtractive closure* of a non-empty subset $X \subseteq S$ is given by

$$\overline{X} := \{s \in S \mid s + x_1 = x_2 \text{ for some } x_1, x_2 \in X\}.$$

If X is a subsemigroup of S , then indeed X is subtractive if and only if $X = \overline{X}$. We call a morphism of Abelian semigroups $f : S \longrightarrow S'$ *subtractive* iff $f(S) \subseteq S'$ is subtractive, equivalently iff

$$f(S) = \{s' \in S' \mid s' + f(s_1) = f(s_2) \text{ for some } s_1, s_2 \in S\}.$$

2.3. A *semiring* is an algebraic structure $(S, +, \cdot, 0, 1)$ consisting of a non-empty set S with two binary operations “+” (addition) and “ \cdot ” (multiplication) satisfying the following conditions:

1. $(S, +, 0)$ is an Abelian monoid with neutral element 0_S ;
2. $(S, \cdot, 1)$ is a monoid with neutral element 1 ;
3. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all $x, y, z \in S$;
4. $0 \cdot s = 0 = s \cdot 0$ for every $s \in S$ (i.e. 0 is *absorbing*).

Let S, S' be semirings. A map $f : S \rightarrow S'$ is said to be a *morphism of semirings* iff for all $s_1, s_2 \in S$:

$$f(s_1 + s_2) = f(s_1) + f(s_2), \quad f(s_1 s_2) = f(s_1) f(s_2), \quad f(0_S) = 0_{S'} \text{ and } f(1_S) = 1_{S'}.$$

The category of semirings is denoted by **SRng**.

- 2.4.** Let $(S, +, \cdot)$ be a semiring. We say that S is
- cancellative* iff the additive semigroup $(S, +)$ is cancellative;
 - commutative* iff the multiplicative semigroup (S, \cdot) is commutative;
 - semifield* iff $(S \setminus \{0\}, \cdot, 1)$ is a commutative group.

Examples 2.5. Rings are indeed semirings. A trivial, but important, example of a *commutative* semiring is $(\mathbb{N}_0, +, \cdot)$ (the set of non-negative integers). Indeed, $(\mathbb{R}_0^+, +, \cdot)$ and $(\mathbb{Q}_0^+, +, \cdot)$ are semifields. A more interesting example is the semi-ring $(\text{ideal}(R), +, \cdot)$ consisting of all ideals of a (not necessarily commutative) ring; this appeared first in the work of *Dedekind* [Ded1894]. On the other hand, for an integral domain R , $(\text{ideal}(R), +, \cap)$ is a semiring if and only if R is a Prüfer domain. Every bounded distributive lattice (R, \vee, \wedge) is a commutative (additively) idempotent semiring. The *additively idempotent* semirings $\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +)$ and $\mathbb{R}_{\min} := (\mathbb{R} \cup \{\infty\}, \min, +)$ play an important role in idempotent and tropical mathematics (e.g. [Lit2007]); the subsemirings $\mathbb{N}_{\max} := (\mathbb{N} \cup \{-\infty\}, \max, +)$ and $\mathbb{N}_{\min} := (\mathbb{N} \cup \{\infty\}, \min, +)$ played an important role in Automata Theory (e.g. [Eil1974], [Eil1976]). The singleton set $S = \{0\}$ is a semiring with the obvious addition and multiplication. In the sequel, we always assume that $0_S \neq 1_S$ so that $S \neq \{0\}$, the *zero semiring*.

2.6. Let S be a semiring. A *right S -semimodule* is an algebraic structure $(M, +, 0_M; \leftarrow)$ consisting of a non-empty set M , a binary operation “+” along with a right S -action

$$M \times S \longrightarrow M, \quad (m, s) \mapsto ms,$$

such that:

1. $(M, +, 0_M)$ is an Abelian monoid with neutral element 0_M ;
2. $(ms)s' = m(ss')$, $(m + m')s = ms + m's$ and $m(s + s') = ms + ms'$ for all $s, s' \in S$ and $m, m' \in M$;

3. $m1_S = m$ and $m0_S = 0_M = 0_M s$ for all $m \in M$ and $s \in S$.

Let M, M' be right S -semimodules. A map $f : M \rightarrow M'$ is said to be a *morphism of right S -semimodules* (or *S -linear*) iff for all $m_1, m_2 \in M$ and $s \in S$:

$$f(m_1 + m_2) = f(m_1) + f(m_2) \text{ and } f(ms) = f(m)s.$$

The set $\text{Hom}_S(M, M')$ of S -linear maps from M to M' is clearly a monoid under addition. The category of right S -semimodules is denoted by \mathbb{S}_S . Similarly, one can define the category of left S -semimodules ${}_S\mathbb{S}$. A right S -semimodule M_S is said to be *cancellative* iff the semigroup $(M, +)$ is cancellative. With $\mathbb{CS}_S \subseteq \mathbb{S}_S$ (resp. ${}_S\mathbb{CS} \subseteq {}_S\mathbb{S}$) we denote the full subcategory of cancellative right (left) S -semimodules.

2.7. Let M be a right S -semimodule. A non-empty subset $L \subseteq M$ is said to be an *S -subsemimodule*, and we write $L \leq_S M$, iff L is closed under “ $+_M$ ” and $ls \in L$ for all $l \in L$ and $s \in S$.

Example 2.8. Every Abelian monoid $(M, +, 0_M)$ is an \mathbb{N}_0 -semimodule in the obvious way. Moreover, the categories **CMon** of commutative monoids and the category $\mathbb{S}_{\mathbb{N}_0}$ of \mathbb{N}_0 -semimodules are isomorphic.

Congruences

2.9. Let M be an S -semimodule. An equivalence relation “ \equiv ” on M is said to be an *S -congruence on M* iff for any $m, m', m_1, m'_1, m_2, m'_2 \in M$ and $s \in S$ we have

$$[m_1 \equiv m'_1 \text{ and } m_2 \equiv m'_2 \Rightarrow [m_1 + m_2 \equiv m'_1 + m'_2]] \text{ and } [m \equiv m' \Rightarrow ms \equiv m's].$$

The set M/\equiv of equivalence classes inherit a structure of an S -semimodule in the obvious way and there is a canonical surjection of S -semimodules $\pi_{\equiv} : M \rightarrow M/\equiv$.

2.10. Let M be an S -semimodule. Every S -subsemimodule $L \leq_S M$ induces two S -congruences on M : the *Bourne relation*

$$m_1 \equiv_L m_2 \Leftrightarrow m_1 + l_1 = m_2 + l_2 \text{ for some } l_1, l_2 \in L;$$

and the *Iizuka relation*

$$m_1 [\equiv]_L m_2 \Leftrightarrow m_1 + l_1 + m' = m_2 + l_2 + m' \text{ for some } l_1, l_2 \in L \text{ and } m' \in M.$$

We call the S -semimodule $M/L := M/_{\equiv_L}$ the *quotient of M by L* or the *factor semimodule* of M by L . One can easily check that $M/L = M/\overline{L}$. If M is cancellative, then L and M/L are cancellative. On the other hand, the S -semimodule $M/[\equiv]_L$ is cancellative.

Proposition 2.11. *The category \mathbb{S}_S and its full subcategory \mathbb{CS}_S have kernels and cokernels, where for any morphism of S -semimodules $f : M \rightarrow N$ we have*

$$\text{Ker}(f) = \{m \in M \mid f(m) = 0\} \text{ and } \text{Coker}(f) = N/f(M).$$

Taking into account the fact that \mathbb{S}_S is a variety (in the sense of Universal Algebra) we have:

Proposition 2.12. ([Tak1982b], [Tak1982c], [TW1989]) *The category of semimodules is*

1. *complete (i.e. has equalizers & products);*
2. *cocomplete (i.e. has coequalizers & coproducts);*
3. *Barr-exact categories* [Bar1971].

Remark 2.13. In [Tak1982c], Takahashi proved that the category of semimodules over a semiring is *c-cocomplete*, which is a *relaxed* notion of cocompleteness which he introduced. However, it was pointed to the author by F. Linton (and other colleagues from the Category List) that such a category is indeed cocomplete in the classical sense since it is a variety.

2.14. As a variety, the category of S -semimodules is regular; in particular, \mathbb{S}_S has a **(RegEpi, Mono)**-factorization structure. Let $\gamma : X \rightarrow Y$ be a morphism of S -semimodules. Then $\text{Im}(\gamma) = \gamma(X)$ and $\text{Coim}(\gamma) = X/f$, where X/f is the quotient semimodule X/\equiv_f given by $x \equiv_f x'$ iff $f(x) = f(x')$. Indeed, we have a canonical isomorphism

$$d_\gamma : \text{Coim}(\gamma) \simeq \text{Im}(\gamma), [x] \mapsto \gamma(x).$$

Remark 2.15. Takahashi defined the *image* of a morphism $\gamma : X \rightarrow Y$ of S -semimodules as $\text{Ker}(\text{coker}(\gamma))$ and the *proper image* as $\gamma(X)$. In fact, $\gamma(X)$ is the *image* of γ in the categorical sense (e.g. [Fai1973, 5.8.7], [EW1987]).

2.16. We call a morphism of S -semimodules $\gamma : M \rightarrow N$:

- subtractive* iff $\gamma(M) \subseteq N$ is subtractive;
- strong* iff $\gamma(M) \subseteq N$ is strong;
- k-uniform* iff for any $x_1, x_2 \in X$:

$$\gamma(x_1) = \gamma(x_2) \implies \exists k_1, k_2 \in \text{Ker}(\gamma) \text{ s.t. } x_1 + k_1 = x_2 + k_2; \quad (3)$$

- i-uniform* iff $\gamma(X) = \overline{\gamma(X)} := \{y \in Y \mid y + \gamma(x_1) = \gamma(x_2) \text{ for some } x_1, x_2 \in X\}$;
- uniform* iff γ is k -uniform and i -uniform;
- semi-monomorphism* iff $\text{Ker}(\gamma) = 0$;
- semi-epimorphism* iff $\overline{\gamma(X)} = Y$;
- semi-isomorphism* iff $\text{Ker}(\gamma) = 0$ and $\overline{\gamma(X)} = Y$.

Remark 2.17. The uniform (k -uniform, i -uniform) morphisms of semimodules were called *regular* (k -regular, i -regular) by Takahashi [Tak1982c]. We think that our terminology avoids confusion sine a regular monomorphism (regular epimorphism) has a different well-established meaning in the language of Category Theory.

Lemma 2.18. *Let $\gamma : X \rightarrow Y$ be a morphism of S -semimodules.*

1. *The following are equivalent:*

- (a) γ is steady;
- (b) $\text{Coker}(\ker(\gamma)) \simeq \text{Coim}(\gamma)$;
- (c) $X/\text{Ker}(\gamma) \simeq \gamma(X)$;
- (d) γ is k -uniform.

2. The following are equivalent:

- (a) γ is costeady;
- (b) $\text{Ker}(\text{coker}(\gamma)) \simeq \text{Im}(\gamma)$;
- (c) $\overline{\gamma(X)} = \gamma(X)$;
- (d) γ is i -uniform (subtractive).

3. The following are equivalent:

- (a) γ is bisteady;
- (b) $\text{Coker}(\ker(\gamma)) \simeq \text{Ker}(\text{Coker}(\gamma))$;
- (c) $X/\text{Ker}(\gamma) \simeq \overline{\gamma(X)}$;
- (d) γ is uniform;

Proof. Notice that the canonical **(Surj, Mono)**-factorization of γ is given by $\gamma : X \xrightarrow{\text{coim}(\gamma)} \gamma(X) \xrightarrow{\text{im}(\gamma)} Y$.

1. By definition, γ is steady iff γ admits a **(Surj, Mono)**-factorization $\gamma = m_1 \circ \text{coker}(\ker(\gamma))$. It follows that γ is steady if and only if $\text{Coker}(\ker(\gamma)) \simeq \text{Coim}(\gamma)$ if and only if $X/\text{Ker}(\gamma) \simeq \gamma(X)$ which is equivalent to γ being k -uniform.
2. By definition γ is costeady if and only if γ admits a **(Surj, Mono)**-factorization $\gamma = \ker(\text{coker}(\gamma)) \circ e_2$. It follows that γ is costeady if and only if $\gamma(X) = \text{Ker}(\text{coker}(\gamma))$. Notice that

$$\begin{aligned} \text{Ker}(\text{coker}(\gamma)) &= \{y \in Y \mid y \equiv_{\gamma(X)} 0\} \\ &= \{y \in Y \mid y + \gamma(x_1) = \gamma(x_2) \text{ for some } x_1, x_2 \in X\} \\ &= \overline{\gamma(X)}. \end{aligned}$$

It follows that γ is costeady if and only if $\gamma(X) = \overline{\gamma(X)}$ which is equivalent to γ being subtractive.

3. This is a combination of “1” and “2”. ■

2.19. Let M be an S -semimodule, $L \leq_S M$ an S -subsemimodule and consider the factor semimodule M/L . Then we have a surjective morphism of S -semimodules

$$\pi_L := M \rightarrow M/L, \quad m \mapsto [m]$$

with

$$\text{Ker}(\pi_L) = \{m \in M \mid m + l_1 = l_2 \text{ for some } l_1, l_2 \in L\} = \overline{L};$$

in particular, $L = \text{Ker}(\pi_L)$ if and only if $L \subseteq M$ is subtractive.

3 Exact Sequences of Semimodules

Throughout this section, S is a ring, an S -semimodule is a right S -semimodule unless otherwise explicitly specified. Moreover, \mathbb{S}_S (\mathbb{CS}_S) denotes the category of (cancellative) right S -semimodules.

The notion of *exact sequences* of semimodules adopted by Takahashi [Tak1981] ($L \xrightarrow{f} M \xrightarrow{g} N$ is exact iff $\overline{f(M)} = \text{Ker}(g)$) seems to be inspired by the definition of exact sequences in Puppe-exact categories. We believe it is inappropriate. The reason for this is that neither $\text{Ker}(\text{coker}(f)) = \overline{f(L)}$ is the appropriate *image* of f nor is $\text{Coker}(\text{ker}(g)) = B/\text{Ker}(g)$ the appropriate *coimage* of g .

Being a Barr-exact category, a natural tool to study exactness in the category of semimodules is that of an *exact fork*, introduced in [Bar1971] and applied to study exact functors between categories of semimodules by Katsov et al. in [KN2011]. However, since the category of semimodules has additional features, one still expects to deal with exact sequences rather than the more complicated exact forks.

In addition to Takahashi's classical definition of exact sequences of semimodules, two different notions of exactness for sequences of semimodules over semirings were introduced recently. The first is due to Patchkoria [Pat2003] ($L \xrightarrow{f} M \xrightarrow{g} N$ is exact iff $f(L) = \text{Ker}(g)$) and the second is due to Patil and Deore [PD2006] ($L \xrightarrow{f} M \xrightarrow{g} N$ is exact iff $\overline{f(L)} = \text{Ker}(g)$ and g is *steady*). Each of these definitions is stronger than Takahashi's notion of exactness and each proved to be more efficient in establishing some nice homological results for semimodules over semirings. However, no clear *categorical* justification for choosing either of these two definitions was provided. A closer look at these definitions shows that they are in fact dual to each other in some sense, and so no it not suitable – in our opinion – to choose one of them and drop the other. This motivated us to introduce in Section one a new notion of exact sequences in general pointed varieties. Applied to categories of semimodules, it turned out that our notion of exact sequences of semimodules is in fact a combination of the two notions of exact sequences of semimodules in the sense of [Pat2003] and [PD2006]. For the sake of completeness, we mention here that there is another notion of exact sequences of semimodules that was introduced in [AM2002]. However, the definition is rather technical and introduced new definitions of *epic* and *monic* morphisms that are different from the classical ones.

As indicated for general varieties in 1.21, the category of semimodules is **(RegEpi, Mono)**-structured, **RegEpi** = **Surj** and **Mono** = **Inj**. We say that a morphism of semimodules $\gamma : X \rightarrow Y$ is *steady* (resp. *costeady*, *bisteady*) iff γ is steady (resp. costeady, bisteady) w.r.t. **(Surj, Inj)**. Moreover, we say that a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of semimodules is *exact* iff it is **(Surj, Inj)**-exact.

Lemma 3.1. *Let*

$$L \xrightarrow{f} M \xrightarrow{g} N \tag{4}$$

be a sequence of S -semimodules with $g \circ f = 0$ and consider the induced morphisms $f' : L \rightarrow \text{Ker}(g)$ and $g'' : \text{Coker}(f) \rightarrow N$.

1. *If f' is an epimorphism, then $\overline{f(L)} = \text{Ker}(g)$.*

2. f' is a regular epimorphism (surjective) if and only if $f(L) = \text{Ker}(g)$ if and only if $\overline{f(L)} = \text{Ker}(g)$ and f is i -uniform.
3. $g'' : \text{Coker}(f) \rightarrow N$ is a monomorphism if and only if $\overline{f(L)} = \text{Ker}(g)$ and g is k -uniform.

Proof. Since $g \circ f = 0$, we have $f(L) \subseteq \overline{f(L)} \subseteq \text{Ker}(g)$.

1. Assume that $f' : L \rightarrow \text{Ker}(g)$ is an epimorphism. Suppose that $\overline{f(L)} \subsetneq \text{Ker}(g)$, so that there exist $m' \in \text{Ker}(g) \setminus \overline{f(L)}$. Consider the S -linear maps

$$L \xrightarrow{\tilde{f}} \text{Ker}(g) \xrightleftharpoons[f_2]{f_1} \text{Ker}(g)/f(L),$$

where $f_1(m) = [m]$ and $f_2(m) = [0]$ for all $m \in \text{Ker}(g)$. For each $l \in L$ we have

$$(f_1 \circ f')(l) = [f(l)] = [0] = (f_2 \circ f')(l).$$

Whence, $f_1 \circ f' = f_2 \circ f'$ while $f_1 \neq f_2$ (since $f_1(m') = [m'] \neq [0] = f_2(m')$; otherwise $m' + f(l_1) = f(l_2)$ for some $l_1, l_2 \in L$ and $m' \in \overline{f(L)}$ which contradicts our assumption). So, f' is not an epimorphism, a contradiction. Consequently, $\overline{f(L)} = \text{Ker}(g)$.

2. Clear.

3. (\Rightarrow) Assume that $g'' : \text{Coker}(f) \rightarrow N$ is a monomorphism. Let $m \in \text{Ker}(g)$, so that $g(m) = 0$. Then $g''([m]) = 0$. Since g'' is a monomorphism, we have $[m] = [0]$ and so $m + f(l) = f(l')$ for some $l, l' \in L$, whence $m \in \overline{f(L)}$. Suppose now that $g(m) = g(m')$ for some $m, m' \in M$. Then $g''([m]) = g''([m'])$ and it follows, by the injectivity of g'' , that $[m] = [m']$ which implies that $m_1 + m_1' = m' + m_1'$ for some $m_1, m_1' \in \overline{f(L)} = \text{Ker}(g)$. So, g is k -uniform.

(\Leftarrow) Assume that $\overline{f(L)} = \text{Ker}(g)$ and that g is k -uniform. Suppose that $g''([m]) = g''([m'])$ for some $m_1, m_2 \in M$. Then $g(m) = g(m')$. Since g is k -uniform, we $m + k = m' + k'$ for some $k, k' \in \text{Ker}(g) = \overline{f(L)}$ and it follows that $[m] = [m']$ (notice that $M/f(L) = M/\overline{f(L)}$). ■

Corollary 3.2. A sequence of semimodules $L \xrightarrow{f} M \xrightarrow{g} N$ is exact if and only if $f(L) = \text{Ker}(g)$ and g is k -uniform.

Remarks 3.3. 1. A morphism of cancellative semimodules $h : X \rightarrow Y$ is an epimorphism in \mathbb{CS}_S if and only if $\overline{h(X)} = Y$. Indeed, if h is an epimorphism, then it follows by Lemma 3.1 that $\overline{h(X)} = Y$ (take $g : Y \rightarrow 0$ as the zero-morphism). On the other hand, assume that $\overline{h(X)} = Y$. Let Z be any cancellative semimodule and consider any S -linear maps

$$X \xrightarrow{h} Y \xrightleftharpoons[h_2]{h_1} Z$$

with $h_1 \circ h = h_2 \circ h$. Let $y \in Y$ be arbitrary. By assumption, $y + h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$, whence

$$h_1(y) + (h_1 \circ h)(x_1) = (h_1 \circ h)(x_2) = (h_2 \circ h)(x_2) = h_2(y) + (h_2 \circ h)(x_1).$$

Since Z is cancellative, we conclude that $h_1(y) = h_2(y)$.

2. Consider the embedding $\iota : \mathbb{N}_0 \hookrightarrow \mathbb{Z}$ in $\mathbb{CS}_{\mathbb{N}_0}$. Indeed, $\overline{\mathbb{N}_0} = \mathbb{Z}$, whence ι is an epimorphism which is not regular.
3. Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence in \mathbb{CS}_S with $g \circ f = 0$. By “1”, the induced morphism $f' : L \rightarrow \text{Ker}(g)$ is an epimorphism if and only if $\overline{f(L)} = \text{Ker}(g)$.

3.4. We call a sequence of S -semimodules $L \xrightarrow{f} M \xrightarrow{g} N$:

proper-exact iff $f(L) = \text{Ker}(g)$;

semi-exact iff $\overline{f(L)} = \text{Ker}(g)$;

quasi-exact iff $\overline{f(L)} = \text{Ker}(g)$ and g is k -uniform;

uniform (resp. *k-uniform*, *i-uniform*) iff f and g are uniform (resp. k -uniform, i -uniform).

3.5. We call a (possibly infinite) sequence of S -semimodules

$$\cdots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \rightarrow \cdots \quad (5)$$

chain complex iff $f_{j+1} \circ f_j = 0$ for every j ;

exact (resp. *proper-exact*, *semi-exact*) iff each partial sequence with three terms $M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2}$ is exact (resp. proper-exact, semi-exact);

uniform (resp. *k-uniform*, *i-uniform*) iff f_j is uniform (resp. k -uniform, i -uniform) for every j .

Definition 3.6. Let M be an S -semimodule.

1. A subsemimodule $L \leq_S M$ is said to be a *uniform (normal) S-subsemimodule* iff the embedding $0 \rightarrow L \xrightarrow{\iota} M$ is uniform (normal).
2. A quotient M/ρ , where ρ is an S -congruence relation on M , is said to be a *uniform (conormal) quotient* iff the surjection $\pi_L : M \rightarrow M/\rho$ is uniform (conormal).

Remark 3.7. Every normal subsemimodule (normal quotient) is uniform.

The following result can be easily verified.

Lemma 3.8. Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of semimodules.

1. Let g be injective.

(a) f is k -uniform if and only if $g \circ f$ is k -uniform.

(b) If $g \circ f$ is i -uniform (uniform), then f is i -uniform (uniform).

(c) Assume that g is i -uniform. Then f is i -uniform (uniform) if and only if $g \circ f$ is i -uniform (uniform).

2. Let f be surjective.

(a) g is i -uniform if and only if $g \circ f$ is i -uniform.

(b) If $g \circ f$ is k -uniform (uniform), then g is k -uniform (uniform).

(c) Assume that f is k -uniform. Then g is k -uniform (uniform) if and only if $g \circ f$ is k -uniform (uniform).

Proof. 1. Let g be injective; in particular, g is k -uniform.

(a) Assume that f is k -uniform. Suppose that $(g \circ f)(l_1) = (g \circ f)(l_2)$ for some $l_1, l_2 \in L$. Since g is injective, $f(l_1) = f(l_2)$. By assumption, there exist $k_1, k_2 \in \text{Ker}(f)$ such that $l_1 + k_1 = l_2 + k_2$. Since $\text{Ker}(f) \subseteq \text{Ker}(g \circ f)$, we conclude that $g \circ f$ is k -uniform. On the other hand, assume that $g \circ f$ is k -uniform. Suppose that $f(l_1) = f(l_2)$ for some $l_1, l_2 \in L$. Then $(g \circ f)(l_1) = (g \circ f)(l_2)$ and so there exist $k_1, k_2 \in \text{Ker}(g \circ f)$ such that $l_1 + k_1 = l_2 + k_2$. Since g is injective, $\text{Ker}(g \circ f) = \text{Ker}(f)$ whence f is k -uniform.

(b) Assume that $g \circ f$ is i -uniform. Let $m \in \overline{f(L)}$, so that $m + f(l_1) = f(l_2)$ for some $l_1, l_2 \in L$. Then $g(m) \in \overline{(g \circ f)(L)} = (g \circ f)(L)$. Since g is injective, $m \in f(L)$. So, f is i -uniform.

(c) Assume that g and f are i -uniform. Let $n \in \overline{(g \circ f)(L)}$, so that $n + g(f(l_1)) = g(f(l_2))$ for some $l_1, l_2 \in L$. Since g is i -uniform, $n \in g(M)$ say $n = g(m)$ for some $m \in M$. But g is injective, whence $m + f(l_1) = f(l_2)$, i.e. $m \in \overline{f(L)} = f(L)$ since f is i -uniform. So, $n = g(m) \in (g \circ f)(L)$. We conclude that $g \circ f$ is i -uniform.

2. Let f be surjective; in particular, f is i -uniform.

(a) Assume that g is i -uniform. Let $n \in \overline{(g \circ f)(L)}$ so that $n + g(f(l_1)) = g(f(l_2))$ for some $l_1, l_2 \in L$. Since g is i -uniform, $n = g(m)$ for some $m \in M$. Since f is surjective, $n = g(m) \in (g \circ f)(L)$. So, $g \circ f$ is i -uniform.

On the other hand, assume that $g \circ f$ is i -uniform. Let $n \in \overline{g(M)}$, so that $n + g(m_1) = g(m_2)$ for some $m_1, m_2 \in M$. Since f is surjective, there exist $l_1, l_2 \in L$ such that $f(l_1) = m_1$ and $f(l_2) = m_2$. Then, $n + (g \circ f)(l_1) = (g \circ f)(l_2)$, i.e. $n \in \overline{(g \circ f)(L)} = (g \circ f)(L) \subseteq g(M)$. So, g is i -uniform.

(b) Assume that $g \circ f$ is k -uniform. Suppose that $g(m_1) = g(m_2)$ for some $m_1, m_2 \in M$. Since f is surjective, we have $(g \circ f)(l_1) = (g \circ f)(l_2)$ for some $l_1, l_2 \in L$. By assumption, $g \circ f$ is k -uniform and so there exist $k_1, k_2 \in \text{Ker}(g \circ f)$ such that $l_1 + k_1 = l_2 + k_2$ whence $m_1 + f(k_1) = m_2 + f(k_2)$. Indeed, $f(k_1), f(k_2) \in \text{Ker}(g)$. i.e. g is k -uniform.

(c) Assume that f and g are k -uniform. Suppose that $(g \circ f)(l_1) = (g \circ f)(l_2)$ for some $l_1, l_2 \in L$. Since g is k -uniform, we have $f(l_1) + k_1 = f(l_2) + k_2$ for some $k_1, k_2 \in \text{Ker}(g)$. But f is surjective; whence $k_1 = f(l'_1)$ and $k_2 = f(l'_2)$ for some

$l_1, l_2 \in L$, i.e. $f(l_1 + l'_1) = f(l_2 + l'_2)$. Since f is k -uniform, $l_1 + l'_1 + k'_1 = l_2 + l'_2 + k'_2$ for some $k'_1, k'_2 \in \text{Ker}(f)$. Indeed, $l'_1 + k'_1, l'_2 + k'_2 \in \text{Ker}(g \circ f)$. We conclude that $g \circ f$ is k -uniform. ■

Remark 3.9. Let $L \leq_S M \leq_S N$ be S -semimodules. It follows directly from the previous lemma that if L is uniform in N , then L is a uniform in M as well. Moreover, if M is uniform in N , then L is uniform in N if and only if L is uniform in M .

Our notion of exactness allows characterization of special classes of morphisms in a way similar to that in homological categories (compare with [BB2004, Proposition 4.1.9], [Tak1981, Propositions 4.4, 4.6], [Gol1999a, Proposition 15.15]):

Proposition 3.10. *Consider a sequence of semimodules*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.$$

1. *The following are equivalent:*

- (a) $0 \longrightarrow L \xrightarrow{f} M$ is exact;
- (b) $\text{Ker}(f) = 0$ and f is steady;
- (c) f is semi-monomorphism and k -uniform;
- (d) f is injective;
- (e) f is a monomorphism.

2. $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$ is semi-exact and f is uniform if and only if $L \simeq \text{Ker}(g)$.

3. $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$ is exact if and only if $L \simeq \text{Ker}(g)$ and g is k -uniform.

4. *The following are equivalent:*

- (a) $M \xrightarrow{\gamma} N \rightarrow 0$ is exact;
- (b) $\text{Coker}(\gamma) = 0$ and γ is costeady;
- (c) γ is semi-epimorphism and i -uniform;
- (d) γ is surjective;
- (e) γ is a regular epimorphism;
- (f) γ is a subtractive epimorphism

5. $L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is semi-exact and g is uniform if and only if $N \simeq \text{Coker}(f)$.

6. $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is exact if and only if $N \simeq \text{Coker}(f)$ and f is i -uniform.

Corollary 3.11. *The following are equivalent:*

- 1. $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is a exact sequence of S -semimodules;

2. $L \simeq \text{Ker}(g)$ and $\text{Coker}(f) \simeq N$;

3. f is injective, $f(L) = \text{Ker}(g)$, g is surjective and $(k-)$ uniform.

In this case, f and g are uniform morphisms.

Remark 3.12. A morphism of semimodules $\gamma : X \longrightarrow Y$ is an isomorphism if and only if $0 \longrightarrow X \longrightarrow Y \longrightarrow 0$ is exact if and only if γ is a uniform bimorphism. The assumption on γ to be uniform cannot be removed here. For example, the embedding $\iota : \mathbb{N}_0 \longrightarrow \mathbb{Z}$ is a bimorphism of commutative monoids (\mathbb{N}_0 -semimodules) which is not an isomorphism. Notice that ι is not i -uniform; in fact $\overline{\iota(\mathbb{N}_0)} = \mathbb{Z}$.

Lemma 3.13. (Compare with [Tak1981, Proposition 4.3.]) *Let $\gamma : X \rightarrow Y$ be a morphism of S -semimodules.*

1. *The sequence*

$$0 \rightarrow \text{Ker}(\gamma) \xrightarrow{\text{ker}(\gamma)} X \xrightarrow{\gamma} Y \xrightarrow{\text{coker}(\gamma)} \text{Coker}(\gamma) \rightarrow 0 \quad (6)$$

is semi-exact. Moreover, (6) is exact if and only if γ is uniform.

2. *We have two exact sequences*

$$0 \rightarrow \overline{\gamma(X)} \xrightarrow{\text{ker}(\text{coker}(\gamma))} Y \xrightarrow{\text{coker}(\gamma)} Y/\gamma(X) \rightarrow 0.$$

and

$$0 \rightarrow \text{Ker}(\gamma) \xrightarrow{\text{ker}(\gamma)} X \xrightarrow{\text{coker}(\text{ker}(\gamma))} X/\text{Ker}(\gamma) \rightarrow 0.$$

Corollary 3.14. (Compare with [Tak1981, Proposition 4.8.]) *Let M be an S -semimodule.*

1. *Let ρ an S -congruence relation on M and consider the sequence of S -semimodules*

$$0 \longrightarrow \text{Ker}(\pi_\rho) \xrightarrow{\iota_\rho} M \xrightarrow{\rho} M/\rho \longrightarrow 0.$$

(a) $0 \rightarrow \text{Ker}(\pi_\rho) \xrightarrow{\iota_\rho} M \xrightarrow{\pi_\rho} M/\rho \rightarrow 0$ *is exact.*

(b) $M/\rho = \text{Coker}(\iota_\rho)$, *whence M/ρ is a normal quotient.*

2. *Let $L \leq_S M$ an S -subsemimodule.*

(a) *The sequence $0 \rightarrow L \xrightarrow{\iota} M \xrightarrow{\pi_L} M/L \rightarrow 0$ is semi-exact.*

(b) $0 \rightarrow \overline{L} \xrightarrow{\iota} M \xrightarrow{\pi_L} M/L \rightarrow 0$ *is exact.*

(c) *The following are equivalent:*

i. $0 \rightarrow L \xrightarrow{\iota} M \xrightarrow{\pi_L} M/L \rightarrow 0$ *is exact;*

ii. $L \simeq \text{Ker}(\pi_L)$;

iii. $0 \longrightarrow L \xrightarrow{\iota} \overline{L} \longrightarrow 0$ *is exact;*

iv. L *is a uniform subsemimodule;*

v. L *is a normal subsemimodule.*

4 Homological lemmas

In this section we prove some elementary diagram lemma for semimodules over semirings. These apply in particular to commutative monoids, considered as semimodules over the semiring of non-negative integers. Recall that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of semimodules is exact iff $f(A) = \text{Ker}(g)$ and g is k -uniform (equivalently, $f(A) = \text{Ker}(g)$ and $g(b) = g(b') \implies b + f(a) = b' + f(a')$ for some $a, a' \in A$).

The following result can be easily proved using *diagram chasing* (compare “2” with [Pat2006, Lemma 1.9]).

Lemma 4.1. *Consider the following commutative diagram of semimodules*

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \\
 \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
 L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \\
 \downarrow & & & & \\
 0 & & & &
 \end{array}$$

and assume that the first and the third columns are exact (i.e. α_1 is surjective and α_3 is injective).

1. Let α_2 be surjective. If the first row is exact, then the second row is exact.
2. Let α_2 be injective. If the second row is exact, then the first row is exact.
3. Let α_2 be an isomorphism. The first row is exact if and only if the second row is exact.

Proof. 1. Let α_2 be surjective and assume that the first row is exact.

- $f_2(L_2) = \text{Ker}(g_2)$.

Notice that $g_2 \circ f_2 \circ \alpha_1 = g_2 \circ \alpha_2 \circ f_1 = \alpha_3 \circ g_1 \circ f_1 = 0$. Since α_1 is an epimorphism, we conclude that $g_2 \circ f_2 = 0$; in particular, $f_2(L_2) \subseteq \text{Ker}(g_2)$. On the other hand, let $m_2 \in \text{Ker}(g_2)$. Since α_2 is surjective, there exists $m_1 \in M_1$ such that $\alpha_2(m_1) = m_2$. Since α_3 is a semi-monomorphism and $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = 0$, we conclude that $g_1(m_1) = 0$. Since the first row is exact, there exists $l_1 \in L_1$ such that $m_1 = f_1(l_1)$. It follows that $m_2 = (\alpha_2 \circ f_1)(l_1) = f_2(\alpha_1(l_1)) \in f_2(L_2)$.

- g_2 is k -uniform.

Suppose that $g_2(m_2) = g_2(m'_2)$. Since α_2 is surjective, there exist $m_1, m'_1 \in M_1$ such that $\alpha_2(m_1) = m_2$ and $\alpha_2(m'_1) = m'_2$. Since α_3 is injective and $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = (g_2 \circ \alpha_2)(m'_1) = (\alpha_3 \circ g_1)(m'_1)$ we have $g_1(m_1) = g_1(m'_1)$.

Since g_1 is k -uniform and $f_1(L_1) = \text{Ker}(g_1)$ there exist $l_1, l'_1 \in L_1$ such that $m_1 + f_1(l_1) = m'_1 + f_1(l'_1)$. It follows that $m_2 + (\alpha_2 \circ f_1)(l_1) = m'_2 + (\alpha_2 \circ f_1)(l'_1)$ whence $m_2 + f_2(\alpha_1(l_1)) = m'_2 + f_2(\alpha_1(l'_1))$. Since $f_2(L_2) \subseteq \text{Ker}(g_2)$, we conclude that g_2 is k -uniform.

2. Let α_2 be injective and assume that the second row is exact.

- $f_1(L_1) = \text{Ker}(g_1)$.

Notice that $\alpha_3 \circ g_1 \circ f_1 = g_2 \circ \alpha_2 \circ f_1 = g_2 \circ f_2 \circ \alpha_1 = 0$. Since α_3 is a monomorphism, we conclude that $g_1 \circ f_1 = 0$, i.e. $f_1(L_1) \subseteq \text{Ker}(g_1)$. Let $m_1 \in \text{Ker}(g_1)$. Then $(g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1) = 0$. Since the second row is exact, there exist $l_2 \in L_2$ such that $f_2(l_2) = \alpha_2(m_1)$. Since α_1 is surjective, there exists $l_1 \in L_1$ such that $\alpha_2(m_1) = f_2(l_2) = f_2(\alpha_1(l_1)) = (\alpha_2 \circ f_1)(l_1)$. Since α_2 is injective, $m_1 = f_1(l_1)$.

- g_1 is k -uniform.

Suppose that $g_1(m_1) = g_1(m'_1)$ for some $m_1, m'_1 \in M_1$. Then we have $(g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1) = (\alpha_3 \circ g_1)(m'_1) = (g_2 \circ \alpha_2)(m'_1)$. Since g_2 is k -uniform and $f_2(L_2) = \text{Ker}(g_2)$, there exist $l_2, l'_2 \in L_2$ such that $\alpha_2(m_1) + f_2(l_2) = \alpha_2(m'_1) + f_2(l'_2)$. Since α_1 is surjective, there exist $l_1, l'_1 \in L_1$ such that $\alpha_2(m_1 + f_1(l_1)) = \alpha_2(m_1) + (f_2 \circ \alpha_1)(l_1) = \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l'_1) = \alpha_2(m'_1 + f_1(l'_1))$. Since α_2 is injective, we have $m_1 + f_1(l_1) = m'_1 + f_1(l'_1)$ and we are done since $f_1(L_1) \subseteq \text{Ker}(g_1)$.

3. This is a combination of “1” and “2”. ■

\mathcal{R} -Homological Categories

It is well-known that the category of groups, despite being non-Abelian (in fact not even Puppe-exact, but semiabelian in the sense of Janelidze et al. [JMT2002]), satisfies the so-called *Short Five Lemma*. It was shown in [BB2004, Theorem 4.1.10] that satisfying this lemma characterizes the so-called *protomodular categories*, whence the *homological categories*, among the pointed regular ones. Inspired by this, we introduce in what follows a notion of (*weak*) relative homological categories with prototype the category of cancellative commutative monoids, or more generally, the categories of cancellative semimodules over semirings.

Definition 4.2. Let \mathfrak{C} be a pointed category and $\mathcal{R} = ((\mathbf{E}, \mathbf{M}); \mathcal{A})$ where (\mathbf{E}, \mathbf{M}) is a factorization structure for \mathfrak{C} and $\mathcal{A} \subseteq \text{Mor}(\mathfrak{C})$. We say that \mathfrak{C} satisfies the *Short \mathcal{R} -Five Lemma* iff for every commutative diagram with (\mathbf{E}, \mathbf{M}) -exact rows and $\alpha_2 \in \mathcal{A}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \longrightarrow & 0 \\ & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \longrightarrow & 0 \end{array}$$

if α_1 and α_3 are isomorphisms, then α_2 is an isomorphism.

Definition 4.3. Let \mathfrak{C} be a category and $\mathcal{R} = ((\mathbf{E}, \mathbf{M}); \mathcal{A})$ where $\mathbf{E}, \mathbf{M}, \mathcal{A} \subseteq \text{Mor}(\mathfrak{C})$. We say that \mathfrak{C} is

1. (\mathbf{E}, \mathbf{M}) -regular iff \mathfrak{C} has finite limits, is (\mathbf{E}, \mathbf{M}) -structured and the morphisms in \mathbf{E} are pullback stable.
2. \mathcal{R} -homological category iff \mathfrak{C} is (\mathbf{E}, \mathbf{M}) -regular and satisfies the Short \mathcal{R} -Five Lemma.

Example 4.4. One recovers the homological categories in the sense of [BB2004] (i.e. those which are pointed, regular and protomodular) as follows: a pointed category \mathfrak{C} is homological iff \mathfrak{C} is \mathcal{R} -homological where $\mathcal{R} = ((\mathbf{RegEpi}, \mathbf{Mono}); \text{Mor}(\mathfrak{C}))$.

Lemma 4.5. Consider the following commutative diagram of semimodules with exact rows

$$\begin{array}{ccccc} L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \\ \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\ L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \end{array}$$

1. We have:

- (a) Let g_1 and α_1 be surjective. If α_2 is injective, then α_3 is injective.
- (b) Let f_2 be injective and α_3 a semi-monomorphism. If α_2 is surjective, then α_1 is surjective.

2. Let f_2 be a semi-monomorphism.

- (a) If α_1 and α_3 are semi-monomorphisms, then α_2 is a semi-monomorphism.
- (b) Let f_1, α_2 be cancellative and f_2 be k -uniform. If α_1 and α_3 are injective, then α_2 is injective.
- (c) If g_1, α_1, α_3 are surjective (and α_2 is i -uniform), then α_2 is a semi-epimorphism (surjective).

Proof. 1. Consider the given commutative diagram.

(a) α_3 is injective.

Suppose that $\alpha_3(n_1) = \alpha_3(n'_1)$ for some $n_1, n'_1 \in N_1$. Since g_1 is surjective, $n_1 = g_1(m_1)$ and $n'_1 = g_1(m'_1)$ for some $m_1, m'_1 \in M_1$. It follows that $(g_2 \circ \alpha_2)(m_1) = (g_2 \circ \alpha_2)(m'_1)$. Since g_2 is k -uniform and $f_2(L_2) = \text{Ker}(g_2)$, there exist $l_2, l'_2 \in L_2$ such that $\alpha_2(m_1) + f_2(l_2) = \alpha_2(m'_1) + f_2(l'_2)$. By assumption, α_1 is surjective and so there exist $l_1, l'_1 \in L_1$ such that $\alpha_1(l_1) = l_2$ and $\alpha_1(l'_1) = l'_2$. It follows that

$$\begin{aligned} \alpha_2(m_1) + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l'_1) \\ \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m'_1) + (\alpha_2 \circ f_1)(l'_1) \\ m_1 + f_1(l_1) &= m'_1 + f_1(l'_1) && (\alpha_2 \text{ is injective}) \\ g_1(m_1) &= g_1(m'_1) && (g_1 \circ f_1 = 0) \\ n_1 &= n'_1 \end{aligned}$$

(b) α_1 is surjective.

Let $l_2 \in L_2$. Since α_2 is surjective, there exists $m_1 \in M_1$ such that $f_2(l_2) = \alpha_2(m_1)$. It follows that $0 = (g_2 \circ f_2)(l_2) = (g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1)$, whence $g_1(m_1) = 0$ (since α_3 is a semi-monomorphism). Since the first row is exact, $m_1 = f_1(l_1)$ for some $l_1 \in L_1$ and so $f_2(l_2) = \alpha_2(m_1) = (\alpha_2 \circ f_1)(l_1) = (f_2 \circ \alpha_1)(l_1)$. Since f_2 is injective, we have $l_2 = \alpha_1(l_1)$.

2. Let f_2 be a semi-monomorphism, *i.e.* $\text{Ker}(f_2) = 0$.

(a) We claim that α_2 is a semi-monomorphism.

Suppose that $\alpha_2(m_1) = 0$ for some $m_1 \in M_1$. Then $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = 0$, whence $g_1(m_1) = 0$ since $\text{Ker}(\alpha_3) = 0$. It follows that $m_1 = f_1(l_1)$ for some $l_1 \in L_1$. So, $0 = \alpha_2(m_1) = (\alpha_2 \circ f_1)(l_1) = (f_2 \circ \alpha_1)(l_1)$, whence $l_1 = 0$ since both f_2 and α_1 are semi-monomorphisms; consequently, $m_1 = f_1(l_1) = 0$.

(b) We claim that α_2 is injective.

Suppose that $\alpha_2(m_1) = \alpha_2(m'_1)$ for some $m_1, m'_1 \in M_1$. Then $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = (g_2 \circ \alpha_2)(m'_1) = (\alpha_3 \circ g_1)(m'_1)$, whence $g_1(m_1) = g_1(m'_1)$ since α_3 is injective. Since g_1 is k -uniform and $\text{Ker}(g_1) = f_1(L_1)$, there exist $l_1, l'_1 \in L_1$ such that $m_1 + f_1(l_1) = m'_1 + f_1(l'_1)$. Then we have

$$\begin{aligned} \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m'_1) + (\alpha_2 \circ f_1)(l'_1) \\ \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l'_1) \\ (f_2 \circ \alpha_1)(l_1) &= (f_2 \circ \alpha_1)(l'_1) && (\alpha_2 \text{ is cancellative}) \\ l_1 &= l'_1 && (f_2 \text{ and } \alpha_1 \text{ are injective}) \\ m_1 + f_1(l'_1) &= m'_1 + f_1(l'_1) && (f_1 \text{ is cancellative}) \\ m_1 &= m'_1 \end{aligned}$$

(c) We claim that α_2 is a semi-epimorphism.

Let $m_2 \in M_2$. Since α_3 and g_1 are surjective, there exists $m_1 \in M_1$ such that $g_2(m_2) = (\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1)$. Since g_2 is k -uniform, $f_2(L_2) = \text{Ker}(g_2)$ and α_1 is surjective, there exist $l_1, l'_1 \in L_1$ such that

$$\begin{aligned} m_2 + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m_1) + (f_2 \circ \alpha_1)(l'_1) \\ m_2 + \alpha_2(f_1(l_1)) &= \alpha_2(m_1 + f_1(l'_1)). \end{aligned}$$

Consequently, $M_2 = \overline{\alpha_2(M_1)}$, *i.e.* α_2 is a semi-epimorphism. If α_2 is i -uniform, then $M_2 = \overline{\alpha_2(M_1)} = \alpha_2(M_1)$, whence α_2 is surjective. ■

Corollary 4.6. *Consider the following commutative diagram of semimodules with exact rows and assume that M_1 and M_2 are cancellative*

$$\begin{array}{ccccccc} L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \longrightarrow & 0 \\ \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ 0 \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \end{array}$$

1. Let α_2 be an isomorphism. Then α_1 is surjective if and only if α_3 is injective.
2. Let α_2 be i -uniform. If α_1 and α_3 are isomorphisms, then α_2 is an isomorphism.

Proposition 4.7. (The Short Five Lemma) *Consider the following commutative diagram of semimodules with M_1, M_2 cancellative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \longrightarrow & 0 \\ & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \longrightarrow & 0 \end{array}$$

Then α_1, α_3 are isomorphisms and α_2 is i -uniform if and only if α_2 is an isomorphism. In particular, the category \mathbb{CS}_S of cancellative right S -semimodules is \mathcal{R} -homological, where $\mathcal{R} = ((\mathbf{Surj}, \mathbf{Inj}); \mathcal{I})$ and \mathcal{I} is the class of i -uniform morphisms.

Lemma 4.8. *Consider the following commutative diagram of semimodules with exact rows*

$$\begin{array}{ccccccccc} U_1 & \xrightarrow{e_1} & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \xrightarrow{h_1} & V_1 \\ \gamma \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \delta \downarrow \\ U_2 & \xrightarrow{e_2} & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \xrightarrow{h_2} & V_2 \end{array}$$

1. Let γ be surjective.
 - (a) If α_1 is injective and α_3 is a semi-monomorphism, then α_2 is a semi-monomorphism.
 - (b) Assume that f_1 and α_2 are cancellative. If α_1 and α_3 are injective, then α_2 is injective.
2. Let δ be a semi-monomorphism. If α_1, α_3 are surjective (and α_2 is i -uniform), then α_2 is a semi-epimorphism (surjective).
3. Let f_1, α_2 be cancellative, γ be surjective and δ be injective. If α_1 and α_3 are isomorphisms, then α_2 is injective and a semi-epimorphism.

Proof. Assume that the diagram is commutative and that the two rows are exact.

1. Let γ be surjective.
 - (a) Assume that α_1 is injective and that α_3 is a semi-isomorphism. We claim that α_2 is a semi-monomorphism.
 Suppose that $\alpha_2(m_1) = 0$ for some $m_1 \in M_1$ so that $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = 0$. Since α_3 is a semi-monomorphism $g_1(m_1) = 0$, whence $m_1 = f_1(l_1)$ for some $l_1 \in L_1$. So, $0 = \alpha_2(m_1) = (\alpha_2 \circ f_1)(l_1) = (f_2 \circ \alpha_1)(l_1)$, whence $\alpha_1(l_1) = (e_2 \circ \gamma)(u_1) = (\alpha_1 \circ e_1)(u_1)$ for some $u_1 \in U_1$ (since γ is surjective and $\text{Ker}(f_2) = e_2(U_2)$). Since α_1 is injective, it follows that $l_1 = e_1(u_1)$ whence $m_1 = f_1(l_1) = (f_1 \circ e_1)(u_1) = 0$.

- (b) Assume that f_1, α_2 are cancellative and α_1, α_3 are injective. We claim that α_2 is injective.

Suppose that $\alpha_2(m_1) = \alpha_2(m'_1)$ for some $m_1, m'_1 \in M_1$. Then $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = (g_2 \circ \alpha_2)(m'_1) = (\alpha_3 \circ g_1)(m'_1)$, whence $g_1(m_1) = g_1(m'_1)$ (notice that α_3 is injective). Since g_1 is k -uniform and $\text{Ker}(g_1) = f_1(L_1)$, there exist $l_1, l'_1 \in L_1$ such that $m_1 + f_1(l_1) = m'_1 + f_1(l'_1)$. Then we have

$$\begin{aligned}
\alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m'_1) + (\alpha_2 \circ f_1)(l'_1) \\
\alpha_2(m'_1) + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l'_1) \\
f_2(\alpha_1(l_1)) &= f_2(\alpha_1(l'_1)) && (\alpha_2 \text{ is cancellative}) \\
\alpha_1(l_1) + k_2 &= \alpha_1(l'_1) + k'_2 && (f_2 \text{ is } k\text{-uniform}) \\
\alpha_1(l_1) + (e_2 \circ \gamma)(u_1) &= \alpha_1(l'_1) + (e_2 \circ \gamma)(u'_1) && (\gamma \text{ is surjective}) \\
\alpha_1(l_1) + (\alpha_1 \circ e_1)(u_1) &= \alpha_1(l'_1) + (\alpha_1 \circ e_1)(u'_1) \\
l_1 + e_1(u_1) &= l'_1 + e_1(u'_1) && (\alpha_1 \text{ is injective}) \\
f_1(l_1) &= f_1(l'_1) && (f_1 \circ e_1 = 0) \\
m_1 + f_1(l_1) &= m'_1 + f_1(l'_1) \\
m'_1 + f_1(l'_1) &= m_1 + f_1(l'_1) \\
m'_1 &= m_1 && (f_1 \text{ is cancellative})
\end{aligned}$$

2. Let δ be a semi-monomorphism. Assume that α_1 and α_3 are surjective. Let $m_2 \in M_2$. Since α_3 is surjective, there exists $n_1 \in N_1$ such that $g_2(m_2) = \alpha_3(n_1)$. It follows that $0 = (h_2 \circ g_2)(m_2) = (h_2 \circ \alpha_3)(n_1) = (\delta \circ h_1)(n_1)$, whence $h_1(n_1) = 0$ since δ is a semi-monomorphism. Since $g_1(M_1) = \text{Ker}(h_1)$, we have $n_1 = g_1(m_1)$ for some $m_1 \in M_1$. Notice that $(g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1) = \alpha_3(n_1) = g_2(m_2)$. Since g_2 is k -uniform, $f_2(L_2) = \text{Ker}(g_2)$ and α_1 is surjective, there exists $l_1, l'_1 \in L_1$ such that

$$\begin{aligned}
m_2 + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m_1) + (f_2 \circ \alpha_1)(l'_1) \\
m_2 + \alpha_2(f_1(l_1)) &= \alpha_2(m_1 + f_1(l'_1)),
\end{aligned}$$

i.e. $m_2 \in \overline{\alpha_2(M_1)}$. Consequently, $M_2 = \overline{\alpha_2(M_1)}$. If α_2 is i -uniform, then $\alpha_2(M) = \alpha_2(M_1) = M_2$, i.e. α_2 is surjective.

3. This is a combination of “1” and “2”. ■

Corollary 4.9. (The Five Lemma) *Consider the following commutative diagram of semi-modules with exact rows and columns and assume that f_1 and α_2 are cancellative*

$$\begin{array}{ccccccccc}
& & & & & & & & 0 \\
& & & & & & & & \downarrow \\
U_1 & \xrightarrow{e_1} & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \xrightarrow{h_1} & V_1 \\
\gamma \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \delta \downarrow \\
U_2 & \xrightarrow{e_2} & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \xrightarrow{h_2} & V_2 \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

1. If α_1 and α_3 are injective, then α_2 is injective.
2. Let α_2 be i -uniform. If α_1 and α_3 are surjective, then α_2 is surjective.
3. Let α_2 be i -uniform. If α_1 and α_3 are isomorphisms, then α_2 is an isomorphism.

The Snake Lemma

One of the basic homological lemmas that are proved usually in categories of modules (e.g. [Wis1991]), or more generally in Abelian categories, is the so called *Kernel-Cokernel Lemma* (*Snake Lemma*). Several versions of this lemma were proved also in non-abelian categories (e.g. *homological categories* [BB2004], *relative homological categories* [Jan2006] and incomplete relative homological categories [Jan2010b]).

Lemma 4.10. *Consider the following commutative diagram with exact columns and assume that the second row is exact.*

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \\
 \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
 L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \\
 \beta_1 \downarrow & & \beta_2 \downarrow & & \beta_3 \downarrow \\
 L_3 & \xrightarrow{f_3} & M_3 & \xrightarrow{g_3} & N_3
 \end{array}$$

1. If f_3 is injective and f_2 is cancellative, then the first row is exact.
2. If g_2, β_1 are surjective, the third row is exact (and g_1 is i -uniform), then g_1 is a semi-epimorphism (surjective).

Proof. Assume that the second row is exact.

1. Notice that $\alpha_3 \circ g_1 \circ f_1 = g_2 \circ \alpha_2 \circ f_1 = g_2 \circ f_2 \circ \alpha_1 = 0$, whence $g_1 \circ f_1 = 0$ since α_3 is a monomorphism. In particular, $f_1(L_1) \subseteq \text{Ker}(g_1)$.

- We claim that $f_1(L_1) = \text{Ker}(g_1)$.

Let $m_1 \in \text{Ker}(g_1)$, so that $g_1(m_1) = 0$. It follows that

$$\begin{aligned}
 (\alpha_3 \circ g_1)(m_1) &= 0 \\
 (g_2 \circ \alpha_2)(m_1) &= 0 \\
 \alpha_2(m_1) &= f_2(l_2) && \text{(2nd row is proper exact)} \\
 0 &= (\beta_2 \circ f_2)(l_2) && (\beta_2 \circ \alpha_2 = 0) \\
 0 &= (f_3 \circ \beta_1)(l_2) \\
 \beta_1(l_2) &= 0 && (f_3 \text{ is a semi-monomorphism}) \\
 l_2 &= \alpha_1(l_1) && \text{(1st column is proper exact)} \\
 f_2(l_2) &= (f_2 \circ \alpha_1)(l_1) \\
 \alpha_2(m_1) &= \alpha_2(f_1(l_1)) \\
 m_1 &= f_1(l_1) && (\alpha_2 \text{ is injective})
 \end{aligned}$$

- We claim that g_1 is k -uniform.

Suppose that $g_1(m_1) = g_1(m'_1)$ for some $m_1, m'_1 \in M_1$. It follows that

$$\begin{aligned}
(\alpha_3 \circ g_1)(m_1) &= (\alpha_3 \circ g_1)(m'_1) \\
(g_2 \circ \alpha_2)(m_1) &= (g_2 \circ \alpha_2)(m'_1) \\
\alpha_2(m_1) + f_2(l_2) &= \alpha_2(m'_1) + f_2(l'_2) \text{ (2nd row is exact)} \\
(\beta_2 \circ f_2)(l_2) &= (\beta_2 \circ f_2)(l'_2) \text{ } (\beta_2 \circ \alpha_2 = 0) \\
(f_3 \circ \beta_1)(l_2) &= (f_3 \circ \beta_1)(l'_2) \\
\beta_1(l_2) &= \beta_1(l'_2) \text{ } (f_3 \text{ is injective}) \\
l_2 + \alpha_1(l_1) &= l'_2 + \alpha_1(l'_1) \text{ (first column is exact)} \\
f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= f_2(l'_2) + (f_2 \circ \alpha_1)(l'_1) \\
f_2(l_2) + (\alpha_2 \circ f_1)(l_1) &= f_2(l'_2) + (\alpha_2 \circ f_1)(l'_1) \\
\alpha_2(m_1) + f_2(l_2) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m_1) + f_2(l'_2) + (\alpha_2 \circ f_1)(l'_1) \\
f_2(l'_2) + \alpha_2(m'_1 + f_1(l_1)) &= f_2(l'_2) + \alpha_2(m_1 + f_1(l'_1)) \text{ } (f_2 \text{ is cancellative}) \\
m'_1 + f_1(l_1) &= m_1 + f_1(l'_1) \text{ } (\alpha_2 \text{ is injective})
\end{aligned}$$

Since $f_1(L_1) \subseteq \text{Ker}(g_1)$, it follows that g_1 is k -uniform.

2. We claim that g_1 is a semi-epimorphism.

Let $n_1 \in N_1$. Let $m_2 \in M_2$ be such that $g_2(m_2) = \alpha_3(n_1)$. Then

$$\begin{aligned}
g_3(\beta_2(m_2)) &= \beta_3(g_2(m_2)) \\
&= (\beta_3 \circ \alpha_3)(m_2) \\
&= 0 \quad (\beta_3 \circ \alpha_3 = 0) \\
\beta_2(m_2) &= f_3(l_3) \quad (3\text{rd row is exact}) \\
&= f_3(\beta_1(l_2)) \quad (\beta_1 \text{ is surjective}) \\
&= \beta_2(f_2(l_2)) \\
m_2 + \alpha_2(m_1) &= f_2(l_2) + \alpha_2(m'_1) \text{ (2nd column is exact)} \\
g_2(m_2) + (g_2 \circ \alpha_2)(m_1) &= (g_2 \circ \alpha_2)(m_1) \quad (g_2 \circ f_2 = 0) \\
\alpha_3(n_1 + g_1(m_1)) &= \alpha_3(g_1(m'_1)) \\
n_1 + g_1(m_1) &= g_1(m'_1) \quad (\alpha_3 \text{ is injective})
\end{aligned}$$

Consequently, $N_1 = \overline{g_1(M_1)}$ ($= g_1(M_1)$ if g_1 is assumed to be i -uniform). ■

Similarly, one can prove the following result.

Lemma 4.11. *Consider the following commutative diagram with exact columns and assume that the second row is exact*

$$\begin{array}{ccccc}
L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \\
\alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \\
\beta_1 \downarrow & & \beta_2 \downarrow & & \beta_3 \downarrow \\
L_3 & \xrightarrow{f_3} & M_3 & \xrightarrow{g_3} & N_3 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

1. If g_1 is surjective and f_3 is i -uniform, then the third row is exact.
2. If f_2, α_3 are injective, α_2 is cancellative and the first row is exact, then f_3 is injective.

Proof. Assume that the second row is exact.

1. Notice that $g_3 \circ f_3 \circ \beta_1 = g_3 \circ \beta_2 \circ f_2 = \beta_3 \circ g_2 \circ f_2 = 0$. Since β_1 is an epimorphism, we have $g_3 \circ f_3 = 0$ (i.e. $f_3(L_3) \subseteq \text{Ker}(g_3)$).

- We claim that $f_3(L_3) = \text{Ker}(g_3)$. Let $m_3 \in \text{Ker}(g_3)$.

Since β_2 is surjective, $m_3 = \beta_2(m_2)$ for some $m_2 \in M_2$. It follows that $0 = (g_3 \circ \beta_2)(m_2) = (\beta_3 \circ g_2)(m_2)$, i.e. $g_2(m_2) \in \text{Ker}(\beta_3) = \alpha_3(N_1)$. We have

$$\begin{aligned}
g_2(m_2) &= \alpha_3(n_1) \\
&= (\alpha_3 \circ g_1)(m_1) && (g_1 \text{ is surjective}) \\
&= (g_2 \circ \alpha_2)(m_1) \\
m_2 + f_2(l_2) &= \alpha_2(m_1) + f_2(l'_2) && (2\text{nd row is exact}) \\
\beta_2(m_2) + (\beta_2 \circ f_2)(l_2) &= (\beta_2 \circ f_2)(l'_2) && (\beta_2 \circ \alpha_2 = 0) \\
m_3 + (f_3 \circ \beta_1)(l_2) &= (f_3 \circ \beta_1)(l'_2)
\end{aligned}$$

We conclude that $\text{Ker}(g_3) = \overline{f_3(L_3)} = f_3(L_3)$.

- We claim that g_3 is k -uniform.

Suppose that $g_3(m_3) = g_3(m'_3)$ for some $m_3, m'_3 \in M_3$. Since β_2 is surjective, there exist $m_2, m'_2 \in M$ such that $\beta_2(m_2) = m_3$ and $\beta_2(m'_2) = m'_3$. Then

$$\begin{aligned}
(g_3 \circ \beta_2)(m_2) &= (g_3 \circ \beta_2)(m'_2) \\
(\beta_3 \circ g_2)(m_2) &= (\beta_3 \circ g_2)(m'_2) \\
g_2(m_2) + \alpha_3(n_1) &= g_2(m'_2) + \alpha_3(n'_1) && (3\text{rd column is exact}) \\
g_2(m_2) + (\alpha_3 \circ g_1)(m_1) &= g_2(m'_2) + (\alpha_3 \circ g_1)(m'_1) && (g_1 \text{ is surjective}) \\
g_2(m_2) + (g_2 \circ \alpha_2)(m_1) &= g_2(m'_2) + (g_2 \circ \alpha_2)(m'_1) \\
m_2 + \alpha_2(m_1) + f_2(l_2) &= m'_2 + \alpha_2(m'_1) + f_2(l'_2) && (2\text{nd row is exact}) \\
\beta_2(m_2) + (\beta_2 \circ f_2)(l_2) &= \beta_2(m'_2) + (\beta_2 \circ f_2)(l'_2) && (\beta_2 \circ \alpha_2 = 0) \\
m_3 + (f_3 \circ \beta_1)(l_2) &= m'_3 + (f_3 \circ \beta_1)(l'_2)
\end{aligned}$$

Since $f_3(L_3) \subseteq \text{Ker}(g_3)$, we conclude that g_3 is k -uniform.

2. We claim that f_3 is injective.

Suppose that $f_3(l_3) = f_3(l'_3)$ for some $l_3, l'_3 \in L_3$. Since β_1 is surjective, there exist

$l_2, l'_2 \in L_2$ such that $\beta_1(l_2) = l_3$ and $\beta_1(l'_2) = l'_3$. Then we have

$$\begin{aligned}
(f_3 \circ \beta_1)(l_2) &= (f_3 \circ \beta_1)(l'_2) \\
(\beta_2 \circ f_2)(l_2) &= (\beta_2 \circ f_2)(l'_2) \\
f_2(l_2) + \alpha_2(m_1) &= f_2(l'_2) + \alpha_2(m'_1) \text{ (2nd column is exact)} \\
(g_2 \circ \alpha_2)(m_1) &= (g_2 \circ \alpha_2)(m'_1) \text{ (} g_2 \circ f_2 = 0 \text{)} \\
(\alpha_3 \circ g_1)(m_1) &= (\alpha_3 \circ g_1)(m'_1) \\
g_1(m_1) &= g_1(m'_1) \text{ (} \alpha_3 \text{ is injective)} \\
m_1 + f_1(l_1) &= m'_1 + f_1(l'_1) \text{ (1st row is exact)} \\
\alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m'_1) + (\alpha_2 \circ f_1)(l'_1) \\
\alpha_2(m_1) + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l'_1) \\
f_2(l_2) + \alpha_2(m_1) + (f_2 \circ \alpha_1)(l_1) &= f_2(l_2) + \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l'_1) \\
f_2(l'_2) + \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l_1) &= f_2(l_2) + \alpha_2(m'_1) + (f_2 \circ \alpha_1)(l'_1) \\
f_2(l'_2 + \alpha_1(l_1)) &= f_2(l_2 + \alpha_1(l'_1)) \text{ (} \alpha_2 \text{ is cancellative)} \\
l'_2 + \alpha_1(l_1) &= l_2 + \alpha_1(l'_1) \text{ (} f_2 \text{ is injective)} \\
\beta_1(l'_2) &= \beta_1(l_2) \text{ (} \beta_1 \circ \alpha_1 = 0 \text{)} \\
l'_3 &= l_3. \blacksquare
\end{aligned}$$

Proposition 4.12. (The Nine Lemma) *Consider the following commutative diagram with exact columns and assume that the second row is exact, α_2, f_2 are cancellative and f_3, g_1 are i -uniform*

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \vdots & & \downarrow & & \downarrow \\
0 & \cdots \rightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \rightarrow 0 \\
& & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
0 & \rightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \rightarrow 0 \\
& & \beta_1 \downarrow & & \beta_2 \downarrow & & \beta_3 \downarrow \\
0 & \rightarrow & L_3 & \xrightarrow{f_3} & M_3 & \xrightarrow{g_3} & N_3 \dashrightarrow 0 \\
& & \downarrow & & \downarrow & & \vdots \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Then the first row is exact if and only if the third row is exact.

Proposition 4.13. (The Snake Lemma) *Consider the following diagram of semimodules in which the two middle squares are commutative and the two middle rows are exact. Assume also that the columns are exact (or more generally that α_1, α_3 are k -uniform and α_2 is*

uniform)

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
& \text{Ker}(\alpha_1) & \xrightarrow{\cdots f_K \cdots} & \text{Ker}(\alpha_2) & \xrightarrow{\cdots g_K \cdots} & \text{Ker}(\alpha_3) & \\
& \downarrow \text{ker}(\alpha_1) & & \downarrow \text{ker}(\alpha_2) & & \downarrow \text{ker}(\alpha_3) & \\
& L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \rightarrow 0 \\
& \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & \\
0 \rightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \\
& \downarrow \text{coker}(\alpha_1) & & \downarrow \text{coker}(\alpha_2) & & \downarrow \text{coker}(\alpha_3) & \\
& \text{Coker}(\alpha_1) & \xrightarrow{\cdots f_C \cdots} & \text{Coker}(\alpha_2) & \xrightarrow{\cdots g_C \cdots} & \text{Coker}(\alpha_3) & \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

1. There exist unique morphisms f_K, g_K, f_C and g_C which extend the diagram commutatively.
2. If f_1 is cancellative, then the first row is exact.
3. If f_C is i -uniform, then the last row is exact.
4. There exists a k -uniform connecting morphism $\delta : \text{Ker}(\alpha_3) \rightarrow \text{Coker}(\alpha_1)$ such that $\text{Ker}(\delta) = g_K(\text{Ker}(\alpha_2))$ and $\delta(\text{Ker}(\alpha_3)) = \text{Ker}(f_C)$.
5. If α_2 is cancellative and g_K is i -uniform, then the following sequence is exact

$$\text{Ker}(\alpha_2) \xrightarrow{\cdots g_K \cdots} \text{Ker}(\alpha_3) \xrightarrow{\delta} \text{Coker}(\alpha_1) \xrightarrow{\cdots f_C \cdots} \text{Coker}(\alpha_2)$$

Proof. 1. The existence and uniqueness of the morphisms f_K, g_K, f_C and g_C is guaranteed by the definition of the (co)kernels and the commutativity of the middle two squares.

2. This follows from Lemma 4.10 applied to the first three rows.
3. This follows from Lemma 4.11 applied to the last three rows.
4. We show first that δ exists and is well-defined.

- We define δ as follows. Let $k_3 \in \text{Ker}(\alpha_3)$. Choose $m_1 \in M_1$ and $l_2 \in L_2$ such that $g_1(m_1) = k_3$ and $f_2(l_2) = \alpha_2(m_1)$; notice that this is possible since g_1 is surjective and $(g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1) = \alpha_3(k_3) = 0$ whence $\alpha_2(m_1) \in \text{Ker}(g_2) = f_2(L_2)$. Define $\delta(k_3) := \text{coker}(\alpha_1)(l_2) = [l_2]$, the coset of $L_2/\alpha_1(L_1)$ which contains l_2 .

- δ is well-defined, i.e. $\delta(k_3)$ is independent of our choice of $m_1 \in M_1$ and $l_2 \in L_2$ satisfying the stated conditions.

Suppose that $g_1(m_1) = k_3 = g_1(m'_1)$. Since the second row is exact, there exist $l_1, l'_1 \in L_1$ such that $m_1 + f_1(l_1) = m'_1 + f_1(l'_1)$. It follows that

$$\begin{aligned} \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m'_1) + (\alpha_2 \circ f_1)(l'_1) \\ f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= f_2(l'_2) + (f_2 \circ \alpha_1)(l'_1) \\ f_2(l_2 + \alpha_1(l_1)) &= f_2(l'_2 + \alpha_1(l'_1)) \\ l_2 + \alpha_1(l_1) &= l'_2 + \alpha_1(l'_1) & (f_2 \text{ is injective}) \\ [l_2] &= [l'_2] \end{aligned}$$

Thus l_2 and l'_2 lie in the same coset of $L_2/\alpha_1(L_1)$, i.e. δ is well-defined.

- Clearly $\overline{g_K(\text{Ker}(\alpha_2))} \subseteq \text{Ker}(\delta)$ (notice that f_2 is a semi-monomorphism). We claim that $\overline{g_K(\text{Ker}(\alpha_2))} = \text{Ker}(\delta)$.

Suppose that $k_3 \in \text{Ker}(\delta)$ for some $k_3 \in \text{Ker}(\alpha_3)$. Let $m_1 \in M_1$ be such that $g_1(m_1) = k_3$ and consider $l_2 \in L_2$ such that $f_2(l_2) = \alpha_2(m_1)$. By assumption, $[l_2] = \delta(k_3) = 0$, i.e. $l_2 + \alpha_1(l_1) = \alpha_1(l'_1)$ for some $l_1, l'_1 \in L_1$. Then we have

$$\begin{aligned} f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= (f_2 \circ \alpha_1)(l'_1) \\ \alpha_2(m_1) + \alpha_2(f_1(l_1)) &= \alpha_2(f_1(l'_1)) \\ m_1 + f_1(l_1) + k_2 &= f_1(l'_1) + k'_2 & (\alpha_2 \text{ is } k\text{-uniform}) \\ k_3 + g_K(k_2) &= g_K(k'_2) & (g_1 \circ f_1 = 0) \end{aligned}$$

Consequently, $\overline{g_K(\text{Ker}(\alpha_2))} = \text{Ker}(\delta)$.

- Notice that for any $k_3 \in \text{Ker}(\alpha_3)$, we have $(f_C \circ \delta)(k_3) = f_C([l_2])$ where $g_1(m_1) = k_3$ and $f_2(l_2) = \alpha_2(m_1)$. It follows that

$$(f_C \circ \delta)(k_3) = f_C(l_2) = [f_2(l_2)] = [\alpha_2(m_1)] = [0].$$

Consequently, $\delta(\text{Ker}(\alpha_3)) \subseteq \text{Ker}(f_C)$. We claim that $\delta(\text{Ker}(\alpha_3)) = \text{Ker}(f_C)$.

Let $[l_2] \in \text{Ker}(f_C)$, i.e. $[f_2(l_2)] = f_C([l_2]) = [0]$, for some $l_2 \in L_2$. Then there exist $m_1, m'_1 \in M_1$ such that $f_2(l_2) + \alpha_2(m_1) = \alpha_2(m'_1)$. By assumption, α_2 is i -uniform, whence there exists $\mathbf{m}_1 \in M_1$ such that $\alpha_2(\mathbf{m}_1) = f_2(l_2)$. It follows that $(\alpha_3 \circ g_1)(\mathbf{m}_1) = (g_2 \circ \alpha_2)(\mathbf{m}_1) = (g_2 \circ f_2)(l_2) = 0$. So, $g_1(\mathbf{m}_1) \in \text{Ker}(\alpha_3)$ and $\delta(g_1(\mathbf{m}_1)) = [l_2]$. Consequently, $\text{Ker}(f_C) = \delta(\text{Ker}(\alpha_3))$.

- We claim that δ is k -uniform.

Suppose that $\delta(k_3) = \delta(k'_3)$ for some $k_3, k'_3 \in \text{Ker}(\alpha_3)$. Let $m_1, m'_1 \in M_1$ and $l_2, l'_2 \in L_2$ be such that $g_1(m_1) = k_3$, $g_1(m'_1) = k'_3$, $\alpha_2(m_1) = f_2(l_2)$ and $\alpha_2(m'_1) = f_2(l'_2)$. By assumption, $[l_2] = [l'_2]$, i.e. $l_2 + \alpha_1(l_1) = l'_2 + \alpha_1(l'_1)$ for some $l_1, l'_1 \in L_1$. Notice that

$$\begin{aligned} f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= f_2(l'_2) + (f_2 \circ \alpha_1)(l'_1) \\ \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m'_1) + (\alpha_2 \circ f_1)(l'_1) \\ m_1 + f_1(l_1) + k_2 &= m'_1 + f_1(l'_1) + k'_2 & (\alpha_2 \text{ is } k\text{-uniform}) \\ g_1(m_1) + g_K(k_2) &= g_1(m'_1) + g_K(k'_2) & (g_1 \circ f_1 = 0) \\ k_3 + g_K(m_1) &= k'_3 + g_K(m_1) \end{aligned}$$

Since $g_K(\text{Ker}(\alpha_2)) \subseteq \text{Ker}(\delta)$ we conclude that δ is k -uniform.

5. If g_K is i -uniform, then we have $\text{Ker}(\delta) = \overline{g_K(\text{Ker}(\alpha_2))} = g_K(\text{Ker}(\alpha_2))$ and it remains only to prove that f_C is k -uniform.

Suppose that $f_C[l_2] = f_C[l'_2]$ for some $l_2, l'_2 \in L_2$. Then there exist $m_1, m'_1 \in M_1$ such that $f_2(l_2) + \alpha_2(m_1) = f_2(l'_2) + \alpha_2(m'_1)$. It follows that

$$\begin{aligned}
(g_2 \circ \alpha_2)(m_1) &= (g_2 \circ \alpha_2)(m'_1) \quad (g_2 \circ f_2 = 0) \\
(\alpha_3 \circ g_1)(m_1) &= (\alpha_3 \circ g_1)(m'_1) \\
g_1(m_1) + k_3 &= g_1(m'_1) + k'_3 \quad (\alpha_3 \text{ is } k\text{-uniform}) \\
g_1(m_1 + \mathbf{m}_1) &= g_1(m'_1 + \mathbf{m}'_1) \quad (g_1 \text{ is surjective}) \\
m_1 + \mathbf{m}_1 + f_1(\mathbf{l}_1) &= m'_1 + \mathbf{m}'_1 + f_1(\mathbf{l}'_1) \quad (2\text{nd row is exact}) \\
\alpha_2(m_1) + \alpha_2(\mathbf{m}_1) + (\alpha_2 \circ f_1)(\mathbf{l}_1) &= \alpha_2(m'_1) + \alpha_2(\mathbf{m}'_1) + (\alpha_2 \circ f_1)(\mathbf{l}'_1) \\
f_2(l'_2) + \alpha_2(m_1) + \alpha_2(\mathbf{m}_1) + (f_2 \circ \alpha_1)(\mathbf{l}_1) &= [f_2(l'_2) + \alpha_2(m'_1)] + \alpha_2(\mathbf{m}'_1) + (f_2 \circ \alpha_1)(\mathbf{l}'_1) \\
f_2(l'_2) + \alpha_2(m_1) + \alpha_2(\mathbf{m}_1) + (f_2 \circ \alpha_1)(\mathbf{l}_1) &= f_2(l_2) + \alpha_2(m_1) + \alpha_2(\mathbf{m}'_1) + (f_2 \circ \alpha_1)(\mathbf{l}'_1) \\
f_2(l'_2) + \alpha_2(\mathbf{m}_1) + (f_2 \circ \alpha_1)(\mathbf{l}_1) &= f_2(l_2) + \alpha_2(\mathbf{m}'_1) + (f_2 \circ \alpha_1)(\mathbf{l}'_1) \quad (\alpha_2 \text{ is cancellative}) \\
f_2(l'_2 + \mathbf{l}_2 + \alpha_1(\mathbf{l}_1)) &= f_2(l_2 + \mathbf{l}_2 + \alpha_1(\mathbf{l}'_1)) \quad (3\text{rd row is exact}) \\
l'_2 + \mathbf{l}_2 + \alpha_1(\mathbf{l}_1) &= l_2 + \mathbf{l}_2 + \alpha_1(\mathbf{l}'_1) \quad (f_2 \text{ is injective}) \\
[l'_2] + [\mathbf{l}_2] &= [l_2] + [\mathbf{l}_2] \\
[l'_2] + \delta(k_3) &= [l_2] + \delta(k'_3)
\end{aligned}$$

Since $\delta(\text{Ker}(\alpha_3)) \subseteq \text{Ker}(f_C)$, we conclude that f_C is k -uniform. ■

Acknowledgments. The author thanks all mathematicians who clarified to him some issues related to the nature of the categories of semimodules and exact sequences or sent to him related manuscripts especially F. Linton, G. Janelidze, Y. Katsov, A. Patchkoria, H. Porst and R. Wisbauer.

References

- [AHS2004] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories; The Joy of Cats* 2004. Dover Publications Edition (2009) (available at: <http://katmat.math.uni-bremen.de/acc>).
- [AM2002] M. R. Adhikari and P. Mukhopadhyay, *Exact sequences of semimodules*, Bull. Calcutta Math. Soc. 94 (1) (2002), 23–32.
- [Bar1971] M. Barr, *Exact categories*, Exact Categories and Categories of Sheaves, Lec. Not. Math. **236**, pp. 1-120, Springer, 1971.
- [Bar2002] M. Barr, *HSP subcategories of Eilenberg-Moore algebras*, Theory Appl. Categ. 10 (18) (2002), 461–468.
- [BB2004] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-Abelian categories*, Mathematics and its Application 566, Kluwer Academic Publishing, (2004).
- [BD2006] S. K. Bhambri and M. Dubey, *Some results on exact sequences of semimodules analogues to module theory*, Soochow J. Math. 32 (4) (2006), 485–498.

- [Bor1994a] F. Borceux, *Handbook of Categorical Algebra. 1, Basic Category Theory*, Cambridge Univ. Press (1994).
- [Bor1994b] F. Borceux, *Handbook of Categorical Algebra. 2, Categories and Structures*, Cambridge Univ. Press (1994).
- [Bou1991] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, Lecture notes in mathematics 1448, Springer 1991, 43–62.
- [BP1969] H.-B. Brinkmann and D. Puppe, *Abelsche und exakte Kategorien, Korrespondenzen*, Lec. Not. Math. 69, Springer (1969).
- [Buc1955] D. A. Buchsbaum, *Exact categories and duality*, Trans. Amer. Math. Soc. 80 (1955), 1–34.
- [Bue2010] T. Bühler, *Exact categories*, Expo. Math. 28 (1) (2010), 1–69.
- [CE1956] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, N. J. (1956).
- [Ded1894] R. Dedekind, *Über die Theorie der ganzen algebraischen Zahlen*. Supplement XI to P. G. Dirichlet, L.: *Vorlesung über Zahlentheorie*. 4 Aufl., Druck und Verlag, Braunschweig (1894).
- [Dur2007] N. Durov, *New Approach to Arakelov Geometry*, Ph.D. dissertation, Universität Bonn – Germany (2007); available as arXiv:0704.2030v1.
- [Eil1974] S. Eilenberg, *Automata, languages, and machines, Vol. A*, Pure and Applied Mathematics 58. Academic Press, New York (1974).
- [Eil1976] S. Eilenberg, *Automata, languages, and machines. Vol. B*, (with two chapters by Bret Tilson), Pure and Applied Mathematics 59, Academic Press, New York-London (1976).
- [EW1987] H. Ehrbar and O. Wyler, *Images in categories as reflections*, Cahiers Topologie Géom. Différentielle Catég. 28 (2) (1987), 143–159.
- [Fai1973] C. Faith, *Algebra: Rings, Modules and Categories: I*, Springer (1973).
- [Gol1999a] J. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, Dordrecht (1999).
- [Gol1999b] J. Golan, *Power Algebras over Semirings. With Applications in Mathematics and Computer Science*, Kluwer Academic Publishers, Dordrecht (1999).
- [Gol2003] J. Golan, *Semirings and Affine Equations over Them*. Kluwer, Dordrecht (2003).
- [Gri1971] P. A. Grillet, *Regular categories*, Exact Categories and Categories of Sheaves, Lec. Not. Math. 236, 121–222, Springer, 1971.

- [Hel1958] A. Heller, *Homological algebra in abelian categories*, Ann. of Math. (2) 68 (1958), 484–525.
- [HW1998] U. Hebisch and H.J. Weinert, *Semirings: algebraic theory and applications in computer science*, World Scientific Publishing Co. (1998).
- [IK2011] S. N. Il'in and Y. Katsov, *On p -Schrier varieties of semimodules*, Comm. Algebra 39, 1491–1501 (2011).
- [Jan2006] T. Janelidze, *Relative homological categories*, Journal of Homotopy and Related Structures 1 (1), 185–194 (2006).
- [Jan2010b] T. Janelidze, *Snake lemma in incomplete relative homological categories*, Theory Appl. Categ. 23 (4) (2010), 76–91.
- [JMT2002] G. Janelidze, L. Márki and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra 168 (2-3) (2002), 367–386.
- [Kat1997] Y. Katsov, *Tensor products and injective envelopes of semimodules over additively regular semirings*, Algebra Colloq. 4 (2) 121–131 (1997).
- [Kat2004a] Y. Katsov, *On flat semimodules over semirings*, Algebra Universalis 51 (2-3), 287–299 (2004).
- [Kat2004b] Y. Katsov, *Toward homological characterization of semirings: Serre's conjecture and Bass's perfectness in a semiring context*, Algebra Universalis 52 (2-3) (2004), 197–214.
- [KTN2009] Y. Katsov, T. G. Nam and N. X. Tuyen, *On subtractive semisimple semirings*, Algebra Colloq. 16 (3) (2009), 415–426.
- [KN2011] Y. Katsov, T. G. Nam, *Morita equivalence and homological characterization of rings*, J. Alg. Appl. 10 (3), 445–473 (2011).
- [KKM2000] M. Kilp, U. Knauer and A. V. Mikhalev, *Monoids, Acts and Categories*, De Gruyter Expositions in Mathematics 29, Walter de Gruyter: Berlin, 2000.
- [KM1997] V. N. Kolokoltsov, V. P. Maslov, *Idempotent analysis and its applications* (with an appendix by Pierre Del Moral), translated from Russian, Mathematics and its Applications 401, Kluwer Academic Publishers Group, Dordrecht (1997).
- [KS1986] W. Kuich and A. Salomaa, *Semirings, Automata, Languages*, Springer-Verlag, Berlin (1986).
- [Lit2007] G. L. Litvinov, *The Maslov dequantization, and idempotent and tropical mathematics: a brief introduction*, J. Math. Sci. (N. Y.) 140 (3) 426–444 (2007).

- [LM2005] G. L. Litvinov and V. P. Maslov (eds.), *Idempotent Mathematics and Mathematical Physics*, Contemp. Math. 377, Amer. Math. Soc., Providence, Rhode Island (2005).
- [LMS1999] G. L. Litvinov, V.P. Maslov and G.B. Shpiz, *Tensor products of idempotent semimodules. An algebraic approach*. Math. Notes 65 (3-4) (1999), 479–489.
- [LMS2001] G. L. Litvinov, V. P. Maslov and G. B. Shpiz, *Idempotent functional analysis. An algebraic approach*, Math. Notes 69, 696–729 (2001).
- [Mac1998] S. Mac Lane, *Categories for the working mathematician*. Second edition. Graduate Texts in Mathematics 5, Springer-Verlag (1998).
- [Mik2006] G. Mikhalkin, *Tropical geometry and its applications*, International Congress of Mathematicians. Vol. II, 827–852, Eur. Math. Soc., Zürich (2006).
- [Mit1965] B. Mitchell, *Theory of Categories*, Academic Press (1965).
- [Pat1998] A. Patchkoria, *Crossed semimodules and Schreier internal categories in the category of monoids*, Georgian Math. J. 5 (6), 575–581 (1998).
- [Pat2000a] A. Patchkoria, *Homology and cohomology monoids of presimplicial semimodules*, Bull. Georgian Acad. Sci. 162 (1) (2000), 9–12.
- [Pat2000b] A. Patchkoria, *Chain complexes of cancellative semimodules*, Bull. Georgian Acad. Sci. 162 (2), 206–208 (2000).
- [Pat2003] A. Patchkoria, *Extensions of semimodules and the Takahashi functor $\text{Ext}_\Lambda(C, A)$* , Homology Homotopy Appl. 5 (1), 387–406 (2003).
- [Pat2006] A. Patchkoria, *On exactness of long sequences of homology semimodules*, J. Homotopy Relat. Struct. 1 (1), 229–243 (2006).
- [Pat2009] A. Patchkoria, *Projective semimodules over semirings with valuations in nonnegative integers*, Semigroup Forum 79 (3) (2009), 451–460.
- [PD2006] K. B. Patil and R. P. Deore, *Some results on semirings and semimodules*, Bull. Calcutta Math. Soc. 98 (1), 49–56 (2006).
- [PL] J. L. Peña and O. Lorscheid, *Mapping \mathbb{F}_1 -land, an overview of geometries over the field with one element*, preprint arXiv:0909.0069 (2009).
- [Pup1962] D. Puppe, *Korrespondenzen in abelschen Kategorien*, Math. Ann. 148 1962 1–30.
- [Qui1973] D. Quillen, *Higher algebraic K-theory I*, in: Lecture Notes in Math. 341, Springer-Verlag, (1973), 85–147.
- [Rai1969] D. A. Raïkov, *Semiabelian categories* (Russian), Dokl. Akad. Nauk SSSR **188** (1969), 1006–1009.

- [R-GST2005] J. Richter-Gebert, B. Sturmfels and T. Theobald, *First steps in tropical geometry*, Idempotent mathematics and mathematical physics, Contemp. Math. 377, Amer. Math. Soc., Providence, RI 289–317, 2005.
- [Sch1972] H. Schubert, *Categories*, Springer Verlag, 1972.
- [Tak1979] Takahashi, M., *A bordism category for the ordinary homology theory*, Math. Sem. Notes Kobe Univ. 7 (3) (1979), 547–572.
- [Tak1981] M. Takahashi, *On the bordism categories. II*. Elementary properties of semi-modules. Math. Sem. Notes Kobe Univ. 9 (2) (1981), 495–530.
- [Tak1982a] M. Takahashi, *On the bordism categories. III*. Functors Hom and for semi-modules. Math. Sem. Notes Kobe Univ. 10 (2) (1982), pp. 551–562.
- [Tak1982b] M. Takahashi, *Extensions of semimodules. I*. Math. Sem. Notes Kobe Univ. 10 (2) (1982), pp. 563–592.
- [Tak1982c] M. Takahashi, *Completeness and c-cocompleteness of the category of semimodules*, Math. Sem. Notes Kobe Univ. 10 (2) (1982), 551–562.
- [Tak1985] Takahashi, M., *On semimodules. III. Cyclic semimodules*. Kobe J. Math. 2 (2) (1985), pp. 131–141.
- [TW1989] M. Takahashi and H.X. Wang, *On epimorphisms of semimodules*. Kobe J. Math. 6(2) (1989), 297–298
- [Van1934] H. S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. 40 (1934), 914–920.
- [Wis1991] R. Wisbauer, *Foundations of Module and Ring Theory, a Handbook for Study and Research*, Gordon and Breach Science Publishers (1991).